Three-dimensional solitary waves in the presence of additional surface effects

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ABSTRACT. – Bifurcations from the quiescent state of three dimensional water wave solutions of a sixth order model equation are analysed. The equation in question is a generalization of the Kadomtsev-Petviashvili equation, and is obtained due to the presence of certain surface effects. These effects are caused either by a surface tension with Bond number close to 1/3, or by an elastic ice-sheet floating on the water surface. The equation describing travelling waves is reduced to a system of ordinary differential equations on a center manifold. Solutions having the form of a solitary wave with damped oscillations, propagating in a channel, are obtained. In the direction transverse to the propagation they satisfy boundary conditions which are either periodic or of Dirichlet type. In the periodic case we find both asymmetric and symmetric waves. In particular, some of these solutions fill a gap in the speeds of the travelling waves where no two-dimensional solitary waves exist. We show that the critical spectra of the linear operators of the model equation and of the full water wave problem are identical. © Elsevier, Paris

1. Introduction

The present paper deals with a three-dimensional (3D) model wave equation, describing wave propagation on a surface of a perfect fluid in the presence of an additional surface pressure. This pressure is caused either by surface tension or by an elastic ice-sheet.

The equation considered here

$$(1.1) \qquad \partial_x \left(\partial_t \eta + \eta \partial_x \eta + s \partial_x^3 \eta + \partial_x^5 \eta \right) + \partial_{yy}^2 \eta = 0, \quad s = \pm 1,$$

is a 3D generalization of the fifth order Korteweg-de Vries equation (FKdV):

$$(1.2) \qquad \partial_t \eta + \eta \partial_x \eta + s \partial_x^3 \eta + \partial_x^5 \eta = 0, \quad s = \pm 1.$$

We derive (1.1) from the full system of Euler equations for long gravity-capillary waves of small amplitude when the Bond number b is close to 1/3, and also for surface water waves in the presence of an elastic ice-plate. In both cases a liquid of finite depth is considered. As a terminology, we say that solutions of (1.1) are three dimensional, though their basic domain is $\mathbb{R}^2$, because they approximate the three dimensional velocity vector field in the Euler system. Similarly, we refer to solutions of (1.2) as two dimensional.

The FKdV equation (for the first time considered by Kawahara, 1972) is known to describe a wide class of wave phenomena in dispersive media. Hunter and Scheurle (1988) and Hărăguş (1996) obtained it as the governing model equation for low surface tension 2D gravity-capillary waves. Marchenko (1988) derived it for nonlinear interfacial waves beneath the elastic ice sheet in a low amplitude-long wave limit. Kakutani and Ono

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(1969) noted that the FKdV equation describes also the propagation of long waves in a cold plasma for angles of inclination of the magnetic vector field near the critical value.

Travelling wave solutions of the FKdV equation have interesting properties. For Bound numbers $b > 1/3$, as well as for elastic plates with large initial tensions (i.e. for $s = -1$) the set of steady wave solutions contains solitary waves of depression and periodic waves. The global existence and the nonlinear orbital stability of the solitary waves in this case have been proved by Buffoni et al. (1966) and Il'ichev and Semenov (1992). Buffoni et al. (1996) showed also that if $b > 1/3$ for a certain range of wave speeds the even homoclinic solution is the unique solution among all functions which satisfy the homoclinic boundary conditions. In the other case, water waves with $b < 1/3$, and elastic plates with low initial tensions (i.e. $s = 1$) the set of steady-wave solutions of the FKdV equation contains generalized solitary waves with non-decreasing ripples (Hunter and Scheurle, 1988; Lombardi, 1997), periodic and quasi-periodic waves, and solitary waves with damped oscillations. The last bifurcate from the non-zero wavenumber where the linear spectrum has a minimum for the phase speed. Solitary waves with non-decreasing periodic ripples are a product of non-linear resonance of long waves with high-wavenumber waves. It was shown by Lombardi (1997) that the amplitude of the smallest ripples is exponentially small. Consequently, a solitary wave becomes unsteady and very slowly decays due to radiation of periodic wave trains. The process of radiation is quasi-stationary (Bakholdin and Il'ichev, 1996; Benilov et al., 1993). In Buffoni et al. (1996) the existence of an infinite set of families of homoclinic solutions of the FKdV equation of finite amplitude for $s = 1$ and $V < 1/4$ (V is the travelling wave speed) was established.

Though it is a model equation, the solution set of the FKdV equation contains all known types of solutions of the full system of Euler equations for gravity-capillary waves (Iooss and Kirchgässner, 1992) for water waves in the presence of the elastic ice-sheet (Il'ichev and Kirchgässner, 1997a), and also for waves in a cold plasma (Il'ichev, 1996). Two-dimensional travelling wave solutions with speeds V bifurcate from the quiescent state either for $V = 0$ or for $V = -1/4$. The bifurcating patterns are of solitary wave type if $V > 0$ (generalized solitary waves), and if $V < -1/4$ (solitary waves with damped oscillations). There is a gap $-1/4 < V < 0$ where no solitary wave solutions bifurcating from the quiescent state exist.

In this paper we look for 3D travelling was solutions of (1.1). We concentrate on solitary waves, i.e. solutions decaying in the diection of propagation (the x-axis), though, as a by-product, we also find solutions which are periodic or quasiperiodic in this direction. In the direction transverse to the propagation, the solutions are assumed to be periodic or to satisfy certain boundary conditions. In particular, we show that it is possible to fill the gap $-1/4 < V < 0$ mentioned above with 3D solitary waves. In fact, in the 3D problem every velocity $V > -1/4$ is a bifurcation point. In order to show the existence of these waves we use a dynamical approach based on a center manifold reduction, in which the direction of propagation is taken as evolutionary variable.

During the last decade such an approach appeared to be a powerful tool in the investigation of certain physical problems involving PDEs in cylindrical domains (i.e. domains with one unbounded spatial coordinate). It has been successfully used e.g. for the water wave problem (Kirchgässner, 1988; Iooss and Kirchgässner, 1992) and for elasticity problems (Mielke, 1988b). However, this method fails if the domain has two or more unbounded coordinates. In (1.1) the basic domain is $\mathbb{R}^2$, so we see no chance to get a complete picture of the set of bounded solutions, even of those with small amplitudes. Some additional assumptions for the solutions of (1.1) are needed.

Hărăguş and Kirchgässner (1995) studied solutions of the Kadomtsev-Petviashvili equation which were periodic in the direction of propagation. The direction transverse to the direction of propagation was the only unbounded direction, and such solutions could be obtained via the center manifold approach. Though these types of solutions play an important role for (1.1) we shall not consider them here (see Section 6). Since one of our aims here is to find bifurcation patterns which can fill the gap $-1/4 < V < 0$ above, and in particular patterns which are localized in x, we keep this direction unbounded, and compact the transverse

direction, i.e. the y-axis. We do this by imposing boundary conditions which can be either periodic, Dirichlet or Neumann boundary conditions.

In the case of waves periodic in the y direction, the bifurcation diagram is a complex one. It is qualitatively similar for $s = 1$ and $s = -1$, so we refer here to the case $s = 1$ (see Section 3 for $s = -1$). A bifurcation occurs when one or more eigenvalues of the linearized problem arrive on the imaginary axis. For a fixed period we find a sequence of bifurcation points $-1/4 < V_0 < V_1 < \ldots < \infty$, and any $V > -1/4$ is a bifurcation point for a sequence of periods $0 < T_0 < T_1 < \ldots < \infty$. In our analysis we shall keep the period fixed and take V as bifurcation parameter. A bifurcation is always due to two pairs of complex conjugate eigenvalues of multiplicity 2 which come to the imaginary axis. The first bifurcation (at V_0) occurs for the lowest periodic mode in the y-direction. The process is repeated for each succeeding mode. The eigenvalues which come to the imaginary axis in pairs then diverge along this axis. In addition to these eigenvalues we find another four, if $V \in (-1/4, 0)$, and two, if $V > 0$, eigenvalues which always stay on the imaginary axis. These eigenvalues result from the 2D bifurcation noted above, from $V_0 = -1/4$ and $V_0 = 0$. Furthermore, for any V, zero is a double eigenvalue due to invariances, but its contribution can be eliminated (see Remark 3.1). Consequently, for the first bifurcation one has six purely imaginary eigenvalues, two of multiplicity four and four simple, for $V_0 \in (-1/4, 0)$, and four such eigenvalues, two of multiplicity four and two simple, for $V_0 > 0$.

Here we shall analyse the following two cases.

Case 1) We replace the periodicity in y by the Dirichlet boundary conditions $\eta_{|y=-l} = \eta_{|y=l} = 0$, i.e. we consider wave motions in a channel with zero wave deviation on its walls. Then all the eigenvalues resulting from the 2D bifurcation are eliminated (to such eigenvalues correspond eigenfunctions which are constant in y so they have to be 0 with the condition above), and the eigenvalues of multiplicity 4 are now of multiplicity 2. We are left with only one pair of double, complex conjugate eigenvalues on the imaginary axis and we are in the case of the 1:1 resonance. From the practical point of view these boundary conditions correspond to certain physically relevant problems of wave motion in channels. We mention here a wave motion of a fluid beneath the ice cover in a channel (river) when the ice cover is rigidly attached to banks, or the capillary waves in a narrow channel under slip conditions.

Case 2) We keep the periodic boundary conditions but we assume $V_0 > 0$. Then at bifurcation we have one pair of complex conjugate eigenvalues of multiplicity four and one pair of complex conjugate simple eigenvalues.

In both cases a center manifold reduction combinated with a normal form analysis is used to described the set of small bounded solutions. We briefly summarize the idea of the reduction method in the following.

Consider a dynamical system

$$(1.3) \qquad\qquad\qquad \dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + F\left(\varepsilon, \mathbf{w}\right)$$

where $\mathbf{w}$ is a vector function, $\mathcal{A}$ a closed linear operator acting in an infinite dimensional Hilbert space X, $F \in C^k\left(\mathbb{R}, X\right)$, for some positive integer k, such that $F\left(0, 0\right) = \partial_\mathbf{w} F\left(0, 0\right) = 0$, ε is a small parameter. For an elliptic system in a cylindridal domain it is the unbounded space variable x which plays the role of the evolutionary variable in (1.3).

Assume that $\mathcal{A}$ has a finite number of purely imaginary eigenvalues. Then, under certain assumptions on the resolvent of $\mathcal{A}$ (cf. Vanderbauwhede and Iooss, 1992; Mielke, 1988a), the system (1.3) can be reduced to a system of ordinary differential equations describing completely the set of small bounded solutions of (1.3). Such solutions are of the form $\mathbf{w} = \mathbf{w}_0 + h\left(\varepsilon, \mathbf{w}_0\right)$, with $\mathbf{w}_0$ solution of the reduced system, living in the space spanned by the eigenvectors of $\mathcal{A}$ corresponding to the imaginary eigenvalues. The function $h\left(\varepsilon, \mathbf{w}_0\right)$ is the reduction function and it has the additional property that it inherits the symmetries of the original equations.

The system we consider here is a reversible system, i.e. it has the symmetry:

$$\mathcal{A}R = -R\mathcal{A}, \qquad F(\varepsilon, R\mathbf{w}) = -RF(\varepsilon, \mathbf{w})$$

where $R : X \to X$ is an isometry ($R^2 = 1$).

For the equation (1.1) we look for travelling wave solutions propagating with speed V in a direction parallel to the x-axis. The x-axis will be the evolutionary variable and the velocity V the bifurcation parameter. Travelling waves are obtained which bifurcate form the quiescent state for speeds $-1/4 < V < \infty$. The wave speeds can be either positive or negative. Negative velocities mean that the wave propagates to the left in the frame moving with the phase speed c of long linear waves, V being actually the difference between the real speed and the speed c. At the bifurcation point the linear operator $\mathcal{A}$ (see (3.2)) has two double purely imaginary eigenvalues in the first case, and one pair of eigenvalues of multiplicity four and one pair of simple purely imaginary eigenvalues in the second case. Due to reversibility we are in the frame of the 1:1 resonance (Iooss and Pérouème, 1993) in the first case. In the second case we perform, for the ten dimensional reduced system, a normal form analysis by using appropriately the O(2) symmetry and the reversibility. Solitary wave solutions will be found in two particular cases.

The paper is organized as follows. In Section 2 the equation (1.1) is derived for both long surface gravity-capillary waves of small amplitude and, for small and long surface waves beneath an elastic sheet. In Section 3 we show that the critical spectrum of $\mathcal{A}$ from (1.1) is the same with the one in the full 3D gravity-capillary problem. This means that near equilibrium the two systems are expected to have the same properties. The case of Dirichlet boundary conditions is considered in Section 4. The center manifold approach is used in Section 4.1 to reduce (1.1) to a four dimensional system of ordinary differential equations. Then, in Section 4.2 the normal form of the reduced system is calculated, and in Section 4.3 the flow on the center manifold is analyzed. We show that a certain coefficient in the normal form of the reduced system is positive and conclude that solitary wave type solutions exist. In a similar manner we treat in Section 5 the case of periodic boundary conditions. The normal form analysis for this case is done in Appendix B. Finally, Section 6 contains a discussion of some further points.

2. Derivation of the governing equation

In this section we derive (1.1) for long surface waves of small amplitude in the presence of (i) an elastic plate, and (ii) surface tension. The first is assumed to obey the equations of the theory of thin plates (Love, 1944).

The Euler system with corresponding additional surface pressure has the form (subscripts denote differentiation with respect to the corresponding variables)

$$
\begin{aligned}
&\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, && -H < z < \eta(x,y,t), \\
&\varphi_z = 0, && z = -H, \\
&\eta_t = \eta_x \varphi_x + \eta_y \varphi_y = \varphi_z, && z = \eta(x,y,t), \\
&\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + g\eta + A\eta_{tt} + B\Delta_{xy}^2 \eta - C\Delta_{xy}\eta = 0.
\end{aligned}
$$

(2.1)

Here $\Delta_{xy} + \partial_{xx}^2 + \partial_{yy}^2$, φ is the velocity potential, η the liquids surface deviation from equilibrium $z = 0$, and x denotes the horizontal unbounded variable. For water beneath an elastic plate we have $A = \rho_i h / \rho_w$, $B = Eh^3 / [12\rho_w(1 - \nu^2)]$, $C = 0$, where ρ_i and ρ_w are the ice and water densities, h and H the ice thickness and water depth, E the Young module and ν the Poisson ratio of the ice. For waves in the presence of surface

tension, $A = B = 0$ and $C = T/\rho_w$, where T is the value of surface tension. In (2.1) we suppress the nonlinear terms corresponding to the additional pressure caused either by surface tensor or by the elastic plate, because they make no contribution to the equation we derive.

In order to determine the relative importance of different terms in the equations above we introduce the following small parameters

$$\varepsilon = \frac{a}{H}, \quad \beta = \frac{C}{g\lambda^2}, \quad \gamma = \frac{B}{g\lambda^4}, \quad \delta = \frac{AH}{\lambda^2}, \quad \mu = \frac{H^2}{\lambda^2},$$

where λ is the characteric wavelength, and a the characteristic wave amplitude. The following dimensionless variables can be defined:

$$t' = \frac{(gH)^{1/2}t}{\lambda}, \quad \varphi' = \frac{(gH)^{1/2}\varphi}{ga\lambda}, \quad \eta' = \frac{\eta}{a}, \quad x' = \frac{x}{\lambda}, \quad y' = \frac{y}{\lambda}, \quad z' = \frac{z}{H}.$$

Then (2.1) is rewritten as follows (omitting the primes):

$$\begin{aligned}
\mu \left(\varphi_{xx} + \varphi_{yy}\right) + \varphi_{zz} &= 0, & -1 < z < \varepsilon\eta \\
\varphi_z &= 0, & z = -1 \\
\eta_t + \varepsilon\eta_x\varphi_x + \varepsilon\eta_y\varphi_y &= \mu^{-1}\varphi_z, & z = \varepsilon\eta \\
\varphi_t + \frac{1}{2}\varepsilon\left(\varphi_x^2 + \varphi_y^2 + \mu^{-1}\varphi_z^2\right) + \eta - \beta\Delta_{xy}\eta + \gamma\Delta_{xy}^2\eta &= 0, & z = \varepsilon\eta.
\end{aligned}$$

(2.2)

The velocity potential can be expanded with respect to the vertical coordinate z:

$$\varphi = \varphi_0 + z\varphi_z^0 + \frac{1}{2}z^2\varphi_{zz}^0 + \frac{1}{6}z^3\varphi_{zzz}^0 + \frac{1}{24}z^4\varphi_{zzzz}^0 + \frac{1}{120}z^5\varphi_{zzzzz}^0 + \frac{1}{720}z^6\varphi_{zzzzzz}^0 + \ldots \quad (2.3)$$

From the first equation in (2.2) we obtain

$$\begin{aligned}
\varphi_{zz}^0 &= -\mu\left(\varphi_{xx}^0 + \varphi_{yy}^0\right), \quad \varphi_{zzz}^0 = -\mu\left[(\varphi_z^0)_{xx} + (\varphi_z^0)_{yy}\right], \\
\varphi_{zzzz}^0 &= \mu^2\left(\varphi_{xxxx}^0 + 2\varphi_{xxyy}^0 + \varphi_{yyyy}^0\right), \\
\varphi_{zzzzz}^0 &= \mu^2\left[(\varphi_z^0)_{xxxx} + 2(\varphi_z^0)_{xxyy} + (\varphi_z^0)_{yyyy}\right], \\
\varphi_{zzzzzz}^0 &= -\mu^3\left(\varphi_{xxxxxx}^0 + 3\varphi_{xxxxyy}^0 + 3\varphi_{xxyyyy}^0 + \varphi_{yyyyyy}^0\right).
\end{aligned}$$

With the help of (2.3), by expanding in power series for small μ up to (but not including) terms of order μ^4, and by using the second equality in (2.2) we derive the expression for φ_z^0:

$$\varphi_z^0 = -\mu\Delta_{xy}\varphi^0 - \frac{\mu^2}{3}\Delta_{xy}^2\varphi^0 - \frac{2}{15}\mu^3\Delta_{xy}^2\varphi^0. \quad (2.4)$$

Substituting (2.4) into the last two equations in (2.2) and neglecting terms of order $\varepsilon\mu$ and higher, we deduce (dropping the superscript ^{0}) the system

$$\begin{aligned}
\eta_t + \varepsilon\eta_x\varphi_x + \varepsilon\eta_y\varphi_y + \varepsilon\eta\Delta_{xy}\varphi + \Delta_{xy}\varphi + \frac{\mu}{3}\Delta_{xy}^2\varphi + \frac{2}{15}\mu^2\Delta_{xy}^2\varphi &= 0 \\
\varphi_t + \frac{\varepsilon}{2}\left(\varphi_x^2 + \varphi_y^2\right) + \eta - \beta\Delta_{xy}\eta + \gamma\Delta_{xy}^2\eta &= 0.
\end{aligned}$$

(2.5)

Next, introduce the unknown functions in the form

(2.6)
$$\eta = \eta_0 + \eta_1 \mu + \eta_2 \mu^2 + O(\mu^3)$$
$$\varphi = \varphi_0 + \varphi_1 \mu + \varphi_2 \mu^2 + O(\mu^3).$$

Neglecting terms of order 3, we look for waves travelling in one direction (to the right), i.e., weakly depending on time in the reference frame moving with the phase speed of the linear wavetrain having an infinite wavelength:

$$\eta = \hat{\eta}(\xi, \zeta, \tau, \mu)$$
$$\hat{\varphi}(\xi, \zeta, \tau, \mu), \quad \xi = x - t, \quad \zeta = \sqrt{\varepsilon} y \quad \tau = \mu^m t, \quad m > 0.$$

(i) *Long waves beneath an ice sheet.* Let $m = 1$, $\delta = \hat{\delta}\mu$, $\gamma = \hat{\gamma}\mu$, $\beta = 0$, $\varepsilon = \hat{\varepsilon}\mu$, where quantities with hat are of order 1, i.e. $E/[12(1 - \nu^2)\rho_w] \sim g\lambda^2 H^2/h^3$. In a real medium this corresponds to the following values of parameters: $h \sim 1m$, $E \sim 10^9 Nm^{-2}$, $H \sim 10m$, $\lambda \sim 100m$, and $a \sim 1m$. Substituting (2.6) into (2.5), we obtain, up to terms of order μ^2,

$$-\hat{\eta}_{0\xi} - \mu\hat{\eta}_{1\xi} + \mu\hat{\eta}_{0\tau} + \hat{\varepsilon}\mu\hat{\eta}_{0\xi}\hat{\varphi}_{0\xi} + \hat{\varepsilon}\mu\hat{\eta}_0\hat{\varphi}_{0\xi\xi} + \hat{\varphi}_{0\xi\xi} + \hat{\varepsilon}\mu\hat{\eta}_{0\zeta\zeta} + \mu\hat{\varphi}_{0\xi\xi} + \frac{\mu}{3}\hat{\varphi}_{0\xi\xi\xi\xi} = 0$$

$$-\hat{\varphi}_{0\xi} - \mu\hat{\varphi}_{1\xi} + \hat{\varphi}_{0\tau} + \frac{1}{2}\hat{\varepsilon}\mu\hat{\varphi}_{0\xi}^2 + \hat{\eta}_0 + \mu\hat{\eta}_1 + \hat{\delta}\mu\hat{\eta}_{0\xi\xi} + \hat{\gamma}\mu\hat{\eta}_{0\xi\xi\xi\xi} = 0.$$

By equating the terms of order 1 and μ we find

(2.7) $\hat{\varphi}_{0\xi} = \hat{\eta}_0,$

and the following equation for $\hat{\eta}_0$ (by using (2.7)):

(2.8)
$$\left[\hat{\eta}_{0\tau} + \frac{3}{2}\hat{\varepsilon}\hat{\eta}_0\hat{\eta}_{0\xi} + \frac{1}{2}\left(\hat{\delta} + \frac{1}{3}\right)\hat{\eta}_{0\xi\xi\xi}\right]_\xi + \hat{\varepsilon}\hat{\eta}_{0\zeta\zeta} = 0.$$

(ii) *Long water waves in the presence of surface tension.* Let $m = 2$, $\beta = \mu(1/3 - \alpha)$, $\alpha = (1/3 - \theta) = \hat{\alpha}\mu$, where $\theta = T/\rho_w g H^2$ is the Bond number, and $\varepsilon = \hat{\varepsilon}\mu^2$. Consequently, our consideration concerns the case when the Bound number θ is close to 1/3. Substituting (2.6) into (2.5) and neglecting terms of order μ^3 we obtain the following evolution equations

$$-\hat{\eta}_{0\xi} - \mu\hat{\eta}_{1\xi} - \mu^2\hat{\eta}_{2\xi} + \mu^2\hat{\eta}_{0\tau} + \hat{\varepsilon}\mu^2\hat{\eta}_{0\xi}\hat{\varphi}_{0\xi} + \hat{\varepsilon}\mu^2\hat{\eta}_0\hat{\varphi}_{0\xi\xi}$$

$$+ \hat{\varphi}_{0\xi\xi} + \hat{\varepsilon}\mu^2\hat{\varphi}_{0\zeta\zeta} + \mu\hat{\varphi}_{1\xi\xi} + \mu^2\hat{\varphi}_{2\xi\xi}\frac{\mu}{3}\hat{\varphi}_{0\xi\xi\xi\xi} + \frac{\mu^2}{3}\hat{\varphi}_{1\xi\xi\xi\xi} + \frac{2}{15}\mu^2\hat{\varphi}_{0\xi\xi\xi\xi\xi\xi} = 0,$$

$$-\hat{\varphi}_{0\xi} - \mu\hat{\varphi}_{1\xi} - \mu^2\hat{\varphi}_{0\tau} + \mu^2\hat{\varphi}_{0\tau} + \frac{1}{2}\hat{\varepsilon}\mu^2\hat{\varphi}_{0\xi}^2 + \hat{\eta}_0 + \mu\hat{\eta}_1 + \mu^2\hat{\eta}_2 - \frac{\mu}{3}\hat{\eta}_{0\xi\xi}$$

$$-\frac{\mu}{3}\hat{\eta}_{1\xi\xi} + \hat{\alpha}\mu^2\hat{\eta}_{0\xi\xi} = 0.$$

Proceeding as in the previous case one obtains

$$\hat{\varphi}_{0\xi} = \hat{\eta}_0, \quad \hat{\eta}_1 = \hat{\varphi}_{1\xi} + \frac{1}{3}\hat{\eta}_{0\xi\xi},$$

and

(2.9)
$$\left[\hat{\eta}_{0\tau} + \frac{3}{2}\hat{\varepsilon}\hat{\eta}_0\hat{\eta}_{0\xi} + \frac{1}{2}\hat{\alpha}\hat{\eta}_{0\xi\xi} + \frac{1}{90}\hat{\eta}_{0\xi\xi\xi\xi\xi}\right]_\xi + \hat{\varepsilon}\hat{\eta}_{\zeta\zeta} = 0.$$

Equations (2.8), (2.9) can be put in the form (1.1) with the help of scaling transformations. Note that when the Bound number θ is less than $\frac{1}{3}$ ($\hat{\alpha} < 0$) the coefficient s in (1.1) equals -1.

3. Comparison of the spectra of the model equation and the full 3D capillary-gravity problem

Travelling waves of (1.1) propagating with velocity V satisfy

(3.1)
$$\partial^2_{yy}\eta - V\partial^2_x\eta + \partial_x\left(\eta\partial_x\eta\right) + s\partial^4_x\eta + \partial^6_x\eta = 0.$$

Assume $V = V_0 + \varepsilon$ where V_0 is some fixed value of the speed, and ε is a small parameter. Then (3.1) may be written in the form (1.3), with

$$\mathbf{w} = (\eta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5)^t, \quad \eta_i = \partial^i_x\eta, \quad i = 1, \ldots, 5,$$

(3.2)
$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\partial^2_{yy} & 0 & V_0 & 0 & -s & 0 \end{pmatrix}, \quad F(\varepsilon, \mathbf{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \varepsilon\eta_2 - \eta^2_1 - \eta\eta_2 \end{pmatrix}.$$

This system has several symmetries. It is invariant under any translation in y, $\tau_a\mathbf{w}(y) = \mathbf{w}(y + a)$, under reflection $S\mathbf{w}(y) = \mathbf{w}(-y)$, and is reversible. We choose as reversibility operator $R = S \circ \tilde{R}$ where

(3.3)
$$\tilde{R} = \text{diag}(1, -1, 1, -1, 1, -1).$$

This choice is justified by the normal form analysis in Appendix B.

We seek solutions of (3.1) which are $4l$-periodic in the y direction. Then the spectrum of $\mathcal{A}$ consists of eigenvalues σ satisfying

(3.4)
$$\sigma^6 + s\sigma^4 - V_0\sigma^2 = \frac{k^2\pi^2}{4l^2},$$

for any k positive integer. Due to reversibility and to the fact that the operator is real, if σ is an eigenvalue then so are $\bar{\sigma}$ and $-\sigma$.

A bifurcation occurs when one or more eigenvalues of $\mathcal{A}$ arrive on the imaginary axis. Therefore, we look for $\sigma = iq$. By the argument above it suffices to consider $q > 0$. Then (3.4) takes the form

(3.5)
$$f(q) = -q^6 + +sq^4 + V_0q^2 = \frac{k^2\pi^2}{4l^2}.$$

Zero is an eigenvalue for any V_0, due to the invariance $\eta \to \eta + const$, $V \to V + const$, and it is double due to reversibility. The eigenvalues are coming to the imaginary axis in pairs (as a consequence of reversibility) when the graph of $f(q)$ touches the horizontal line, $\frac{k^2\pi^2}{4l^2}$ (see *Fig.* 1).

REMARK 3.1. – The eigenvalue zero of multiplicity 2 increases the dimension of the center manifold by 2. However, both these 2 additional dimensions can be eliminated by using the identities

(3.6)
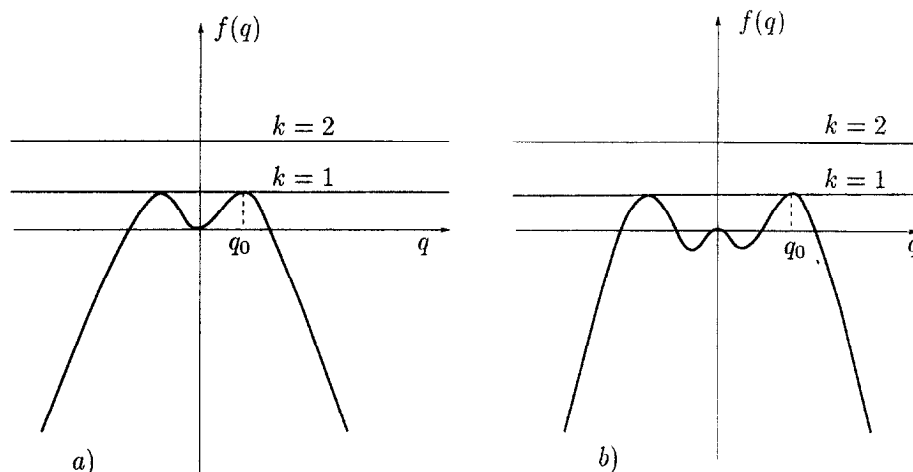$$\int^{2l}_{-2l} (\eta_5 + s\eta_3 + \eta\eta_1 - V\eta_1)\, dy = 0,$$

Fig. 1. – The behaviour of the polynomial $f(q)$ for $s = 1$: a) $V_0 > 0$, b) $-1/4 < V_0 < 0$.

and

$$(3.7) \qquad \int_{-2l}^{2l} \left(\eta_4 + s\eta_2 + \frac{1}{2}\eta^2 - V\eta \right) dy = const,$$

where the constant in the last equality is set to 0 which is the case for solitary waves. These equalities are obtained by integrating (3.1) with respect to y over the period.

If $s = 1$ and $V_0 \leq -1/4$ the polynomial $f(q)$ is negative, so no bifurcations are possible in this case. The same is true if $s = -1$, $V_0 < 0$. For $s = -1$, $V_0 > 0$ the polynomial $f(q)$ behaves itself as in Figure 1a, so the dynamics of critical eigenvalues is qualitatively similar to that for $s = 1$, $V_0 > 0$. Therefore, we restrict our considerations to the case $s = 1$ described in Figure 1.

From (3.5) we deduce that the first bifurcation occurs when

$$(3.8) \qquad V_0 = 3q_0^4 - 2q_0^2 \quad \text{and} \quad 2q_0^6 - q_0^4 = \frac{\pi^2}{4l^2},$$

where q_0 is the largest root of the second equation in (3.8). Furthermore, we deduce

$$q_0^2 > \frac{1}{2}, \quad V_0 > -\frac{1}{4}.$$

For $l \in (0, \infty)$, V_0 varies in the interval $(-1/4, \infty)$. More precisely, for every $V_0 \in (-1/4, \infty)$, there exists a period, determined by l, for which V_0 is a bifurcation point. For $V_0 = -1/4$ and $V_0 = 0$, $l = \infty$ one obtains the 2D solutions (independent of y solutions), bifurcating from the quiescent state. For $V_0 < 0$ there are four, and for $V_0 > 0$ – two additional eigenvalues which always lie on the imaginary axis. These eigenvalues are the result of the "influence" of 2D solutions, which bifurcate from $V_0 = -1/4$, or $V_0 = 0$ for $l = \infty$. In general this influence on the bifurcation diagram cannot be neglected. The dynamics of eigenvalues coming to the imaginary axis for both $V_0 < 0$ and $V_0 > 0$ and $k = 1$ is shown in Figure 2. Note that two new pairs come to the imaginary axis (i.e. a new bifurcation takes place) when the graph of the polynomial $f(q)$ first intersects the next line $k^2\pi^2/4l^2$, parallel to the q-axis. The eigenvalues then diverge along the imaginary axis. The multiplicity of these eigenvalues depends on the boundary conditions we impose in y and we discuss it later.
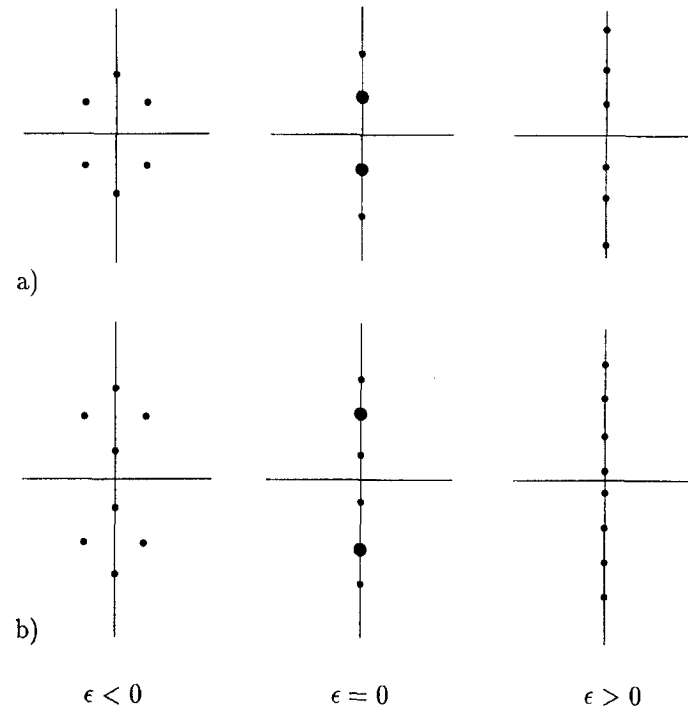
Fig. 2. – Dynamics of critical eigenvalues: a) $V_0 > 0$, b) $-1/4 < V_0 < 0$. Zero eigenvalues are suppressed.

The linearized equations for travelling wave solutions of the full gravity-capillary problem are obtained from (2.1):

(3.9)
$$
\begin{aligned}
\varphi_{xx} + \varphi_{yy} + \varphi_{zz} &= 0, & -1 < z < 0, \\
\varphi_z &= 0, & z = -1, \\
\eta_x &= \varphi_z, & z = 0, \\
\varphi_x + \lambda\eta - b\Delta_{xy}\eta &= 0, & z = 0.
\end{aligned}
$$

Here λ is the inverse square of the Froude number and b is the Bond number

$$
\lambda = \frac{gh}{V^2}, \quad b = \frac{T}{\rho_w H V^2},
$$

where V denotes the speed of the travelling wave. The equations (3.9) are written in the dimensionless form. We use the scaling

$$
(x, y, z, \eta) \rightarrow \frac{1}{H}(x, y, z, \eta), \quad \varphi \rightarrow \frac{\varphi}{VH}.
$$

The system (3.9) is of the form

$$
\dot{\mathbf{v}} = \mathcal{A}\mathbf{v},
$$

(where dot denotes differentiation with respect to x) with

$$\mathbf{v} = (\varphi, u, \eta, \phi)^t, \quad u = \partial_x \varphi, \quad \phi = \partial_x \eta,$$

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\partial^2/\partial y^2 - \partial^2/\partial z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/b & \lambda b - \partial^2/\partial y^2 & 0 \end{pmatrix}$$

In the domain of definition of the linear operator $\mathcal{A}$ we include the (linear) boundary conditions:

$$\phi = \varphi_z|_{z=0}, \quad \varphi_z|_{z=1} = 0.$$

The eigenvalue problem $\mathcal{A}\mathbf{f} = \sigma\mathbf{f}$ for $4l$-periodic in y functions yields the dispersion relation

$$\left(p - \frac{\Omega^2}{p}\right)\cosh p - (\lambda + bp^2)\sinh p = 0,$$

where

$$\Omega = \frac{k\pi}{2l}, \quad p = \sqrt{q^2 + \Omega^2}, \quad q = -i\sigma.$$

Note that, as for the model equation (1.3), zero is always an eigenvalue of multiplicity two, due to the gauge invariance $\varphi \to \varphi + const$. Then eigenvalues come to the imaginary axis in pairs on the curves in the (b, λ) plane given by

$$b = \left(1 + \frac{\Omega^2}{p^2}\right)\frac{\coth p}{2p} - \left(p - \frac{\Omega^2}{p^2}\right)\frac{\sinh^{-2} p}{2p}$$

$$\lambda = \left(p - 3\frac{\Omega^2}{p}\right)\frac{\coth p}{2} + \left(p^2 - \Omega^2\right)\frac{\sinh^{-2} p}{2}.$$

In the parameter space (b, λ) we find, for fixed l, infinitely many bifurcation curves for $k = 0, 1, 2, \ldots$, which accumulate on the semiaxis $(b = 0, \lambda > 0)$, as $k \to \infty$. The first curve for $k = 0$ is that obtained in the 2D and is independent of l. If l varies on $(0, \infty)$ these curves fill the region left of the curve for $k = 0$. In other words, at any point (b, λ) left of the 2D bifurcation curve we can find a bifurcation curve for the 3D problem for a suitable l. The transition of eigenvalues coming to the imaginary axis is shown in Figure 3.

The equation (3.1) is derived under the assumption that $b \sim 1/3$. Values $\lambda > 1$ corresponds to $V_0 < 0$, and $\lambda < 1$ to $V_0 > 0$. Recall that $b > 1/3$ corresponds to $s = -1$ and $b < 1/3$ to $s = 1$ in the equation (3.1). From Figure 3 it can be seen that if $\lambda < 1$ the behaviour of critical eigenvalues for (b, λ) along the bifurcation curve $k = 1$ corresponds to the dynamics of eigenvalues pictured in Figure 2a, i.e. for the case $V_0 > 0$, $s = 1$. This is also true for $V_0 > 0$, $s = -1$. For $\lambda > 1$, $b < 1/3$ (i.e. $-1/4 < V_0 < 0$, $s = 1$ in (3.1)) we also have the full correspondence of bifurcation diagrams shown in Figure 3 and Figure 2b. No bifurcations occur if $\lambda > 1$, $b > 1/3$, as for $V_0 < 0$, $s = -1$ in equation (3.1).

We conclude that the same kind of bifurcations occur in the full system and in the model equation. The same still holds when the periodic boundary conditions in y are replaced by other types of boundary conditions, e.g. Dirichlet or Neumann boundary conditions, periodic boundary conditions in x.

The first bifurcation of 3D patterns which are $4l$-periodic in y occurs along the bifurcation curve for $k = 1$ (see *Fig.* 3). At the bifurcations we find 3 pairs of complex conjugate purely imaginary eigenvalues if $\lambda > 1$, and 2 such pairs if $\lambda < 1$, without counting the double 0 eigenvalue. In both cases, only one of these pairs is due to the 3D bifurcation at $k = 1$. The corresponding eigenvalues are, in the periodic case, of multiplicity 4 (cf. also Bridges, 1994; Loginov and Kuznetsov, 1996). It is the result of the collision of 2 pairs of eigenvalues $\pm s_1 + is_2$ and $\pm s_1 - is_2$ each of multiplicity 2. The other purely imaginary eigenvalues result from the previous
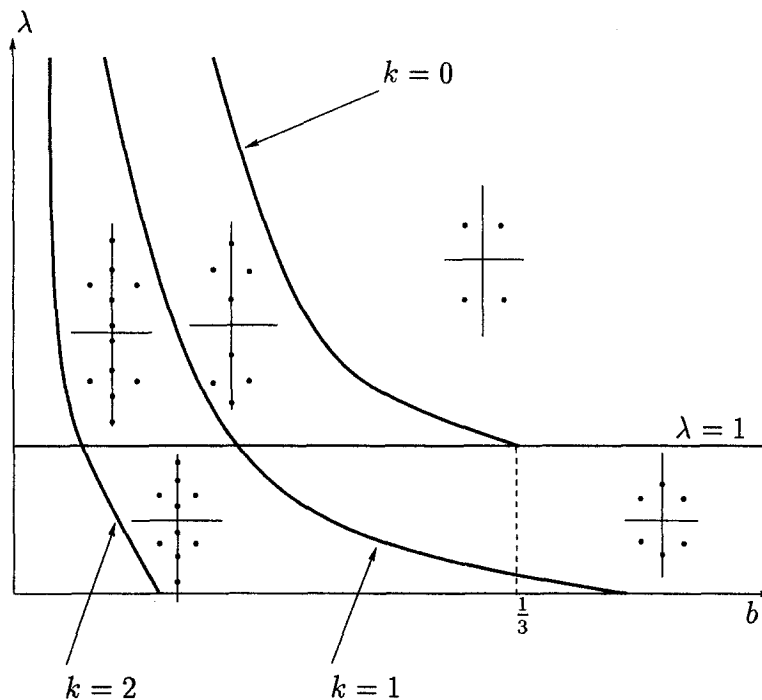
Fig. 3. – Transition of eigenvalues through the imaginary axis in the spectral σ-plane for $\lambda > 0$. The double zero eigenvalue is suppressed.

2D bifurcation and are all simple. This is the case also for the model equation. It is this first bifurcation that we shall study in the sequel.

4. Dirichlet boundary conditions

We showed in the previous section that the equation (3.1) describing travelling wave solutions of (1.1) is of the form

$$(4.1) \qquad \dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + F(\varepsilon, \mathbf{w}),$$

with $\mathbf{w}$, $\mathcal{A}$ and F defined by (3.2). We also found that a center manifold which is 10- or 12-dimensional can be obtained to describe small bounded solutions of (4.1) which are $4l$-periodic in y. In this section we shall treat the easier case when the periodic boundary conditions are replaced by the Dirichlet-type boundary conditions

$$(4.2) \qquad \eta_{|y=-l} = \eta_{|y=l} = 0.$$

We show that bounded solutions of (4.1) satisfying (4.2) are described by a reduced system of ordinary differential equations on a four dimensional center manifold. For this we use a version of the center manifold reduction, cf. Vanderbauwhede and Iooss (1992), and the normal form approach for the 1:1 resonance, cf. Iooss and Pérouème (1993).

We add the conditions (4.2) in the domain of definition of the linear operator $\mathcal{A}$. The consequence is that all eigenvalues having constant eigenvectors (independent of y) are eliminated, and the multiplicity of the other eigenvalues is divided by 2. In particular, the eigenvalues on the imaginary axis obtained for $k = 0$, i.e.

the ones resulting from the intersection of the graph of $f(q)$ with the real axis (see *Fig.* 1), are suppressed. Consequently, we are left with only one pair of double eigenvalues on the imaginary axis, hence with a 4-dimensional center manifold. Note that the eigenvalues eliminated in this way are the ones resulting from the previous 2D bifurcation.

4.1. REDUCTION

Define the spaces

$$\mathcal{X}_l = H_0^{5/3} \times H_0^{4/3} \times H_0^1 \times H_0^{2/3} \times H^{1/3} \times H^0,$$

and

$$\mathcal{Y}_l = H_0^2 \times H_0^{5/3} \times H_0^{4/3} \times H_0^1 \times H_0^{2/3} \times H^{1/3},$$

where H^s denotes the Sobolev space $H^s(-l, l)$, and H_0^s is the closure of $\mathcal{D}(-l, l)$ in $H^s(-l, l)$.

The linear operator $\mathcal{A}$ is closed in $\mathcal{X}_l$ with dense domain $\mathcal{Y}_l$, and since the embedding $\mathcal{Y}_l \subset \mathcal{X}_l$ is compact, $\mathcal{A}$ has compact resolvent. Hence, the spectrum of $\mathcal{A}$ consists only of discrete eigenvalues with no finite point of accumulation. The eigenvalues σ of $\mathcal{A}$ satisfy (3.4) for $k = 1, 2, \ldots$. From the previous results we conclude that for a given wave length l, there exists a first bifurcation point V_0, and that for V close to V_0 the critical spectrum of $\mathcal{A}$ is like that in Figure 4.

We perform the reduction for V close to V_0, so for small $\varepsilon = V - V_0$. Note that due to reversibility (3.3) we are in the case of the 1:1 resonance (cf. Dias and Iooss, 1993; Iooss and Pérouème, 1993).

The generalized eigenvectors of $\mathcal{A}$, associated with the critical eigenvalues $\pm i q_0$, satisfy

$$\mathcal{A}\phi_0 = iq_0\phi_0, \quad \mathcal{A}\phi_1 = iq_0\phi_1 + \phi_0,$$
$$\mathcal{A}\overline{\phi_0} = -iq_0\overline{\phi_0}, \quad \mathcal{A}\overline{\phi_1} = iq_0\overline{\phi_1} + \overline{\phi_0}.$$

A simple calculation yields

$$\phi_0 = (1, iq_0, -q_0^2, -iq_0^3, q_0^4, iq_0^5)^t \cos\left(\frac{\pi}{2l}y\right)$$
$$\phi_1 = (0, 1, 2iq_0, -3q_0^2, -4iq_0^3, 5q_0^4)^t \cos\left(\frac{\pi}{2l}y\right).$$

They satisfy also the reversibility conditions

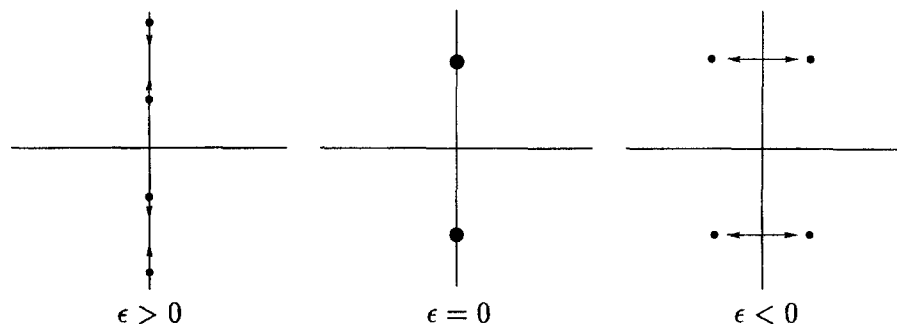$$R\phi_0 = \overline{\phi_0}, \quad R\phi_1 = -\overline{\phi_1}.$$

Fig. 4. – Transition of eigenvalues passing through the imaginary axis in the complex σ-plane (1:1 resonance bifurcation), $k = 1$.

The inequality

$$\|fg\|_{H^0} \le C\|f\|_{H^0}\|g\|_{H^s},$$

which holds for any $s > 1/2$, shows that the nonlinear term F is smooth as a map $F : \mathbb{R} \times \mathcal{X}_l \to \mathcal{X}_l$. Then the version of the center manifold reduction in Vanderbauwhede and Iooss (1992) can be applied, provided the next lemma holds.

LEMMA 4.1. – *Denote by W the closure in $\mathcal{X}_l$ of the range of F. Then there exist $C > 0$ and $\hat{q} > 0$ such that for any $q \in \mathbb{R}$, $|q| > \hat{q}$ the following inequality holds:*

(4.3)
$$\|(\mathcal{A} - iq)^{-1}\|_{W \to \mathcal{X}_l} \le \frac{C}{|q|}.$$

The proof of this lemma is given in Appendix A.

Then we can apply the center manifold reduction theorem, and obtain that all small bounded solutions of (4.1) are of the form

(4.4)
$$\mathbf{w} = A\phi_0 + B\phi_1 + \overline{A\phi_0} + \overline{B\phi_1} + \Phi\left(\varepsilon, A, B, \overline{A}, \overline{B}\right),$$

where Φ consists of higher order terms in A, B, and ε. We take

$$\Phi\left(0, A, B, \overline{A}, \overline{B}\right) = \left(A^2\Phi_{2000} + c.c\right) + |A|^2\Phi_{1100} + \dots.$$

Moreover, the amplitudes A and B satisfy the reduced system

(4.5)
$$A_r = iq_0 A + B + f\left(\varepsilon, A, B, \overline{A}, \overline{B}\right)$$
$$B_r = iq_0 B + g\left(\varepsilon, A, B, \overline{A}, \overline{B}\right).$$

4.2. NORMAL FORM

In this section we compute the coefficients of the normal form of the reduced system (4.5). For this we follow Dias and Iooss (1993).

The system (4.5) can be put in normal form, which for the 1:1 resonance in the presence of reversibility reads

(4.6)
$$A_r = iq_0 A + B + iAP\left[\varepsilon, |A|^2, \frac{1}{2}i\left(A\overline{B} - \overline{A}B\right)\right]$$
$$B_r = iq_0 B + iBP\left[\varepsilon, |A|^2, \frac{1}{2}i\left(A\overline{B} - \overline{A}B\right)\right] + AQ\left[\varepsilon, |A|^2, \frac{1}{2}i\left(A\overline{B} - \overline{A}B\right)\right],$$

where P and Q are polynomials in their arguments with real coefficients. Set

$$P\left(\varepsilon, u, K\right) = p_1\varepsilon + p_2 u + p_3 K + O\left((|\varepsilon| + |u| + |K|^2\right)$$
$$Q\left(\varepsilon, u, K\right) = q_1\varepsilon - q_2 u + q_3 K + O\left((|\varepsilon| + |u| + |K|^2\right).$$

Note that the system (4.6) has two first integrals

$$K = \frac{i}{2}\left(A\overline{B} - \overline{A}B\right), \quad H = |B|^2 - \int_0^{|A|^2} Q\left(\varepsilon, s, K\right) ds.$$

The coefficient q_1 is easy to calculate. It is related to the eigenvalues of the linearization of (4.6)

$$\sigma = iq_0 + iP(\varepsilon, 0, 0) \pm \sqrt{Q(\varepsilon, 0, 0)},$$

so

$$\sigma = iq_0 \pm \sqrt{q_1 \varepsilon} + ip_1 \varepsilon + O(|\varepsilon|^{3/2}).$$

For the case in question $q_1 < 0$, and we obtain

$$q_1 = \frac{1}{4(1 - 3q_0^2)}.$$

The coefficient q_2 is determined via the expression

$$q_2 = -\langle 2N_2(\phi_0, \Phi_{1100}) + 2N_2(\overline{\phi_0}, \Phi_{2000}), \phi_1^* \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(-l, l)$, and ϕ_1^* is the eigenvector of the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ verifying $\mathcal{A}^* \phi_1^* = -iq_0 \phi_1^*$ and $\langle \phi_1, \phi_1^* \rangle = 1$. An elementary calculation yields

$$\phi_1^* = r(q_0)\left(iq_0^3(2q_0^2 - 1), -q_0^2(2q_0^2 - 1), -iq_0(1 - q_0^2), 1 - q_0^2, -iq_0, 1\right)^t \cos\left(\frac{\pi}{2l} y\right),$$

with

$$r(q_0) = \frac{1}{4lq_0^2(3q_0^2 - 1)}.$$

The coefficients Φ_{1100} and Φ_{2000} satisfy the equations

$$\mathcal{A}\Phi_{2000} + N_2(\phi_0, \phi_0) = 2iq_0 \Phi_{2000},$$
$$\mathcal{A}\Phi_{1100} + 2N_2(\phi_0, \overline{\phi_0}) = 0.$$

We obtain

$$\Phi_{1100} = 0, \quad \Phi_{2000} = \left(1, 2iq_0, -4q_0^2, -8iq_0^3, 16q_0^4, 32iq_0^5\right)^t f_1$$

$$f_1 = c_1 \cosh \omega y + c_2 \cos^2\left(\frac{\pi}{2l} y\right) + c_3$$

$$c_1 = \frac{2q_0^2 - 1}{12q_0^2(5q_0^2 - 1)(13q_0^2 - 2)} \operatorname{sech}(\omega l), \quad c_2 = \frac{1}{6q_0^2(1 - 5q_0^2)}$$

$$c_3 = \frac{2q_0^2 - 1}{12q_0^2(1 - 5q_0^2)(13q_0^2 - 2)}, \quad \omega = 2q_0^2 \sqrt{13q_0^2 - 2}.$$

To compute the coefficient q_2 the symbolic package *Mathematica 2.2* was used. We find

$$q_2 = \frac{8l^3\omega^3 + 8l\omega\pi^2 - 43l^3\omega^3 q_0^2 - 43l\omega\pi^2 q_0^2 - 2\pi^2 \tanh\omega l + 4\pi^2 q_0^2 \tanh\omega l}{96\omega l q_0^2 (l^2\omega^2 + \pi^2)(1 - 5q_0^2)(3q_0^2 - 1)(13q_0^2 - 2)}.$$

We conclude that q_2 is always positive, is a decreasing function of q_0, and has the asymptotics

$$q_2 \to \frac{1}{12}, \quad q_0 \to \frac{1}{\sqrt{2}},$$

$$q_2 \sim \frac{43}{13 \cdot 15 \cdot 96 q_0^6}, \quad q_0 \to \infty.$$

(Recall that $q_0 > 1/\sqrt{2}$). The shape of q_2 is given in Figure 5.
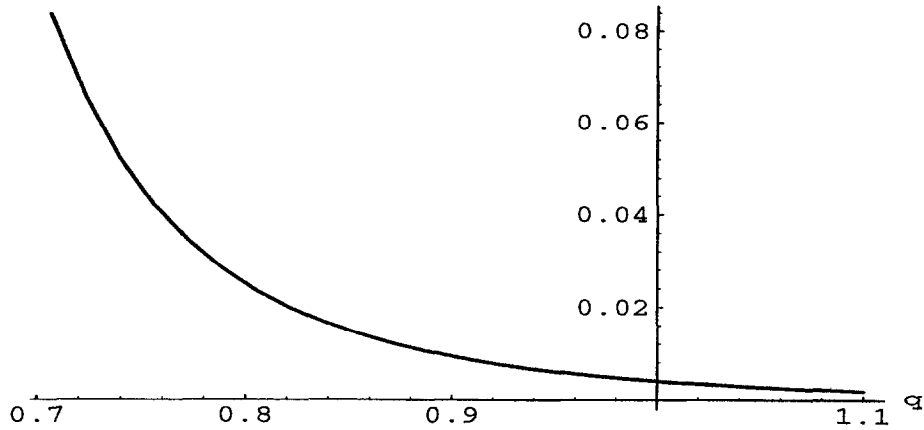
Fig. 5. – The shape of the coefficient q_2 versus q_0.

4.3. SOLITARY WAVE SOLUTIONS

In this section we show that the reduced system possesses for small ε a homoclinic solution to which corresponds a solitary wave solution of (3.1). We use the results of existence of homoclinic solutions in reversible 1:1 resonance vector fields by Iooss and Pérouème (1993).

Consider the transformation

$$A = r_0 \exp i \left(q_0 x + \theta_0\right), \quad B = r_1 \exp i \left(q_0 x + \theta_1\right).$$

In new variables the system of equations (4.6) takes the form

$$(4.7) \qquad \begin{aligned} \left(\frac{du_0}{dx}\right)^2 &= 4 \left\{u_0 \left[G\left(\varepsilon, u_0, K\right) + H\right] - K^2\right\} \\ \frac{d\left(\theta_1 - \theta_0\right)}{dt} &= -K \left(u_0 u_1\right)^{-1} \left[u_0 Q\left(\varepsilon, u_0, K\right) + G\left(\varepsilon, u_0, K\right) + H\right], \end{aligned}$$

where $u_0 = r_0^2$ and $u_1 = r_1^2$. The steady solutions of (4.7), which correspond to periodic solutions of (4.6), are given by the double roots of the polynomial

$$(4.8) \qquad f\left(u_0\right) = u_0 \left[G\left(\varepsilon, u_0, K\right) + H\right] - K^2.$$

Scaling as

$$H = \varepsilon^2 h, \quad u_0 = |\varepsilon| v, \quad K = |\varepsilon|^{3/2} k$$

we obtain

$$f = |\varepsilon|^3 g\left(v, h, k\right) + O\left(|\varepsilon|^{7/2}\right),$$

where

$$g\left(v, h, k\right) = vh + q_1 \operatorname{sign} \varepsilon \, v^2 - \frac{q_2}{2} v^3 - k^2.$$

Double roots of (4.8) to the required order are thus given by double roots of the polynomial $g(v, h, k)$. They lie on the curves in the (h, k) plane given parametrically by

$$(4.9) \qquad k = \pm\sqrt{q_2 v^3 - \text{sign}\,\varepsilon q_1 v^2}, \quad h = \frac{3}{2}\,q_2 v^2 - 2\,\text{sign}\,\varepsilon q_1 v.$$

For $q_2 > 0$ all bounded solutions live in the interior of the set $g(v, h, k) = 0$. The point $(0, 0)$ corresponds to the homoclinic solution given by the formula

$$u_0\,(x) = \frac{2\varepsilon q_1}{q_2}\,\text{sech}^2\,(\sqrt{\varepsilon q_1}x) + O\,(|\varepsilon|^{3/2}).$$

To this homoclinic solution corresponds a solitary wave solution of (3.1) with (4.2). This solution is obtained by reversing the reduction procedure.

THEOREM 4.2. – *Assume* $V_0 > -1/4$. *There exists* $\varepsilon_0 < 0$ *such that for any* $\varepsilon \in (\varepsilon_0, 0)$ *the equation* (3.1) *with* $V = V_0 + \varepsilon$ *has a solitary wave solution*

$$\eta\,(x, y) = \pm 2\sqrt{\frac{2\varepsilon q_1}{q_2}}\,\text{sech}\,(\sqrt{\varepsilon q_1}x)\cos(q_0 x)\cos\left(\frac{\pi}{2l}\,y\right) + O\,(|\varepsilon|^{3/2}).$$

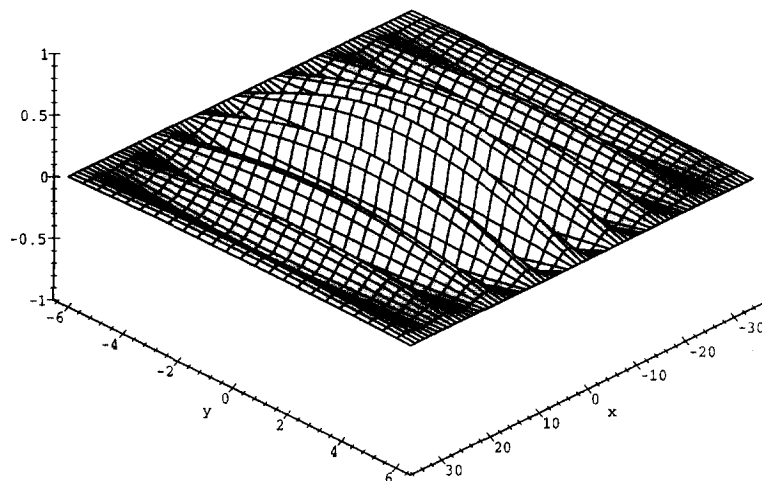In Figure 6 the form of this solitary wave solution is shown.

Fig. 6. – Solitary wave of elevation for $q_0 = 0.75$, $\varepsilon = -0.01$.

Besides the homoclinic solution above the reduced system possesses also periodic and quasi-periodic solutions. They correspond to solutions of (3.1) which are periodic and quasi-periodic in x. For a complete description of these solutions we refer to Dias and Iooss (1993), Iooss and Pérouème (1993)).

5. Periodic case

In this section we assume $V_0 > 0$. We seek solutions of (4.1) which are $4l$-periodic in y. Section 3 we showed that in this case the linear operator $\mathcal{A}$ has, at the bifurcation point, two pairs of complex conjugate purely imaginary eigenvalues, one of eigenvalues of multiplicity four and one of simple eigenvalues. Denote the simple
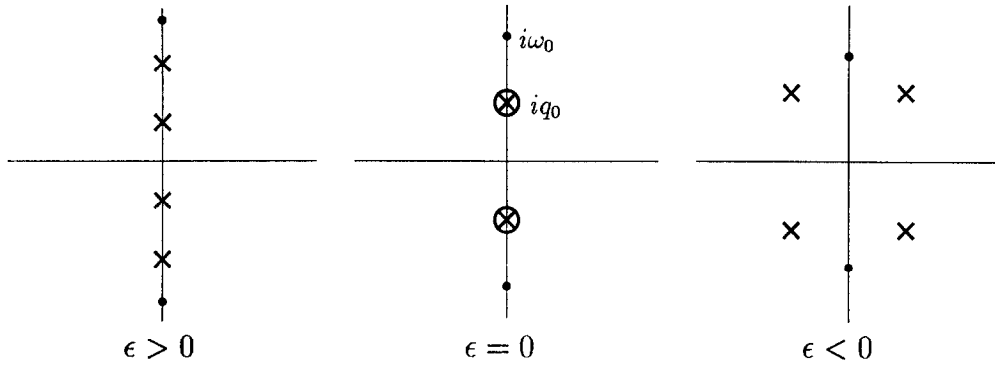
Fig. 7. – Critical eigenvalues in the periodic case, for $V = V_0 + \varepsilon$, $V_0 > 0$. The double eigenvalue zero is omitted.

eigenvalues by $\pm i\omega_0$ and those of multiplicity four by $\pm i q_0$. Then $f(q_0) = \pi^2/4l^2$, and $f(\omega_0) = 0$, where f the function defined in (3.5). The dynamics of these critical eigenvalues for $V = V_0 + \varepsilon$ close is shown in Figure 7.

The justification of the reduction in this case follows exactly that in Section 4 so we will not repeat it here. The only change is that the space $\mathcal{X}_l$ will now contain functions which are $4l$-periodic in their arguments.

5.1. REDUCED SYSTEM

We show in this section that after the reduction we obtain a 12–dimensional reduced system and that two of these dimensions can be eliminated as in Remark 3.1.

We calculate first the (generalized) eigenvectors associated to the purely imaginary eigenvalues. We use the notation

$$E(\sigma) = (1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5)^t,$$

for $\sigma \in \mathbb{C}$, and

$$E'(\phi) = (0, 1, 2\sigma, 3\sigma^2, 4\sigma^3, 5\sigma^4)^t.$$

The eigenvalue 0 is double and has the generalized eigenvectors

$$e_0 = E(0), \quad e_1 = E'(0), \quad \text{with} \quad \mathcal{A}e_0 = 0, \quad \mathcal{A}e_1 = e_0.$$

The eigenvalues $\pm i\omega_0$ are simple and the associated eigenvectors are

$$f_0 = E(i\omega_0), \quad \overline{f}_0 = E(-i\omega_0), \quad \text{with} \quad \mathcal{A}f_0 = i\omega_0 f_0, \quad \mathcal{A}\overline{f}_0 = -i\omega_0\overline{f}_0.$$

Finally, the eigenvalues $\pm i q_0$ are of multiplicity four, and we find to each eigenvalue two eigenvectors and two generalized eigenvectors

$$\phi_0 = \frac{1}{2} E(iq_0) e^{i\frac{\pi}{2l}y}, \quad \psi_0 = \frac{1}{2} E'(iq_0) e^{i\frac{\pi}{2l}y}, \quad \mathcal{A}\phi_0 = iq_0\phi_0, \quad \mathcal{A}\psi_0 = iq_0\psi_0 + \phi_0,$$

$$\phi_1 = \frac{1}{2} E(iq_0) e^{-i\frac{\pi}{2l}y}, \quad \psi_1 = \frac{1}{2} E'(iq_0) e^{-i\frac{\pi}{2l}y}, \quad \mathcal{A}\phi_1 = iq_0\phi_1, \quad \mathcal{A}\psi_1 = iq_0\psi_1 + \phi_1,$$

$$\overline{\phi}_0 = \frac{1}{2} E(-iq_0) e^{-i\frac{\pi}{2l}y}, \quad \overline{\psi}_0 = \frac{1}{2} E'(-iq_0) e^{-i\frac{\pi}{2l}y}, \quad \mathcal{A}\overline{\phi}_0 = -iq_0\overline{\phi}_0, \quad \mathcal{A}\overline{\psi}_0 = -iq_0\overline{\psi}_0 + \overline{\phi}_0,$$

$$\overline{\phi}_1 = \frac{1}{2} E(-iq_0) e^{i\frac{\pi}{2l}y}, \quad \overline{\psi}_1 = \frac{1}{2} E'(-iq_0) e^{i\frac{\pi}{2l}y}, \quad \mathcal{A}\overline{\phi}_1 = -iq_0\overline{\phi}_1, \quad \mathcal{A}\overline{\psi}_1 = -iq_0\overline{\psi}_1 + \overline{\phi}_1$$

The reduction theorem shows that all small enough bounded solutions of (3.1) which are $4l$-periodic in y are of the form

(5.1)
$$\mathbf{w} = ae_0 + be_1 + A_0\phi_0 + B_0\psi_0 + A_1\phi_1 + B_1\psi_1 + Cf_0 + c.c.$$
$$+ \Phi\left(\varepsilon, a, b, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right),$$

where by $c.c.$ we denote the complex conjugates of the previous terms, and Φ is at least of order 2 in its arguments. By substituting (5.1) into (3.1) we obtain a reduced system for the amplitudes $a, b, A_0, B_0, A_1, B_1, C$.

We show now that by imposing the conditions (3.6) and (3.7) we can write a, b as functions of the other amplitudes, so we can eliminate them from the reduced system. A substitution of $\mathbf{w}$ given by (5.1) into (3.6) and (3.7) yields

(5.2)
$$-4lV_0 a + f_1\left(\varepsilon, a, b, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right) = 0,$$

and

(5.3)
$$-4lV_0 b + f_2\left(\varepsilon, a, b, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right) = 0,$$

where f_1 and f_2 are at least quadratic in their arguments. Then by applying the implicit function theorem one can easily deduce

$$a = f_a\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right),$$

and

$$b = f_b\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right),$$

and f_a, f_b are at least quadratic in their arguments. It follows then that we can eliminate a and b and find a reduced system for only A_0, B_0, A_1, B_1, C. Due to the structure of the critical spectrum of $\mathcal{A}$ the reduced system has the form

(5.4)
$$A_{0x} = iq_0 A_0 + B_0 + F_0\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right)$$
$$B_{0x} = iq_0 B_0 + G_0\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right)$$
$$A_{1x} = iq_0 A_1 + B_1 + F_1\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right)$$
$$B_{1x} = iq_0 B_1 + G_1\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right)$$
$$C_x = i\omega_0 C + H\left(\varepsilon, A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C}\right).$$

5.2. Normal form

In this section we compute the normal form of the reduced system (5.4).

We see first how the symmetries τ_a, S, and the reversibility R act on the reduced system. We have

$$\tau_a \phi_0 = e^{i\frac{\pi}{2l}a}\phi_0, \quad \tau_a \psi_0 = e^{i\frac{\pi}{2l}a}\psi_0, \quad \tau_a \phi_1 = e^{-i\frac{\pi}{2l}a}\phi_1, \quad \tau_a \psi_1 = e^{-i\frac{\pi}{2l}a}\psi_1, \quad \tau_a f_0 = f_0,$$

so the induced symmetry is

$$\tau_a^* : \begin{pmatrix} A_0 \to e^{i\frac{\pi}{2i}a}A_0 \\ B_0 \to e^{i\frac{\pi}{2i}a}B_0 \\ A_1 \to e^{-i\frac{\pi}{2i}a}A_1 \\ B_1 \to e^{-i\frac{\pi}{2i}a}B_1 \\ C \to C \end{pmatrix}.$$

For the reflection S we find

$$S\phi_0 = \phi_1, \quad S\psi_0 = \psi_1, \quad Sf_0 = f_0,$$

hence

$$S^* : \begin{pmatrix} A_0 \to A_1 \\ B_0 \to B_1 \\ A_1 \to A_0 \\ B_1 \to B_0 \\ C \to C \end{pmatrix}.$$

Finally, the reversibility yields

$$R^* : \begin{pmatrix} A_0 \to \overline{A}_0 \\ B_0 \to -\overline{B}_0 \\ A_1 \to \overline{A}_1 \\ B_1 \to -\overline{B}_1 \\ C \to \overline{C} \end{pmatrix},$$

since

$$R\phi_0 = \overline{\phi}_0, \quad R\psi_0 = -\overline{\psi}_0, \quad R\phi_1 = \overline{\phi}_1, \quad R\psi_1 = -\overline{\psi}_1, \quad Rf_0 = \overline{f}_0.$$

The reduced system (5.4) is invariant under the action of τ^*, S^*, and its right hand side anticommutes with R^*. The same holds for the normal form. Then, by the results in Appendix B, the normal form of (5.4) is

$$
\begin{aligned}
A_{0,r} &= iq_0 A_0 + B_0 + iA_0 P\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) + o\left(|\mathbf{A}|^n\right) \\
B_{0,r} &= iq_0 B_0 + iB_0 P\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) + A_0 Q\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) + o\left(|\mathbf{A}|^n\right) \\
(5.5) \qquad A_{1,r} &= iq_0 A_1 + B_1 + iA_1 P\left(\varepsilon, u_3, u_4, u_1, u_2, u_5, \overline{\tilde{u}_6}\right) + o\left(|\mathbf{A}|^n\right) \\
B_{1,r} &= iq_0 B_1 + iB_1 P\left(\varepsilon, u_3, u_4, u_1, u_2, u_5, \overline{\tilde{u}_6}\right) + A_1 Q\left(\varepsilon, u_3, u_4, u_1, u_2, u_5, \overline{\tilde{u}_6}\right) + o\left(|\mathbf{A}|^n\right) \\
C_r &= i\omega_0 C + iCR\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) + o\left(|\mathbf{A}|^n\right)
\end{aligned}
$$

with P, Q, R polynomials of order n in their arguments with real coefficients (depending on ε), except for the coefficients of the monomials involving $\tilde{u}_6$. We have used the notation

$$u_1 = A_0\overline{A}_0, \quad u_2 = \frac{i}{2}\left(A_0\overline{B}_0 - \overline{A}_0 B_0\right), \quad u_3 = A_1\overline{A}_1, \quad u_4 = \frac{i}{2}\left(A_1\overline{B}_1 - \overline{A}_1 B_1\right).$$

$$u_5 = C\overline{C}, \quad \tilde{u}_6 = \frac{i}{2}\left(u_1 A_1\overline{B}_1 - u_3\overline{A}_0 B_0\right).$$

and $\mathbf{A} = \left(A_0, B_0, A_1, B_1, C\right)^t$.

The polynomials P, Q, R are of the form

$$P\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) = p_1\varepsilon + p_2 u_1 + p_3 u_2 + p_4 u_3 + p_5 u_4 + p_6 u_5 + \mathcal{O}\left(\left(|\varepsilon| + |\mathbf{A}|^2\right)^2\right),$$

$$Q\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) = q_1\varepsilon - q_2 u_1 + q_3 u_2 + q_4 u_3 + q_5 u_4 + q_6 u_5 + \mathcal{O}\left(\left(|\varepsilon| + |\mathbf{A}|^2\right)^2\right).$$

$$R\left(\varepsilon, u_1, u_2, u_3, u_4, u_5, \tilde{u}_6\right) = r_1\varepsilon + r_2 u_1 + r_3 u_2 + r_4 u_3 + r_5 u_4 + r_6 u_5 + \mathcal{O}\left(\left(|\varepsilon| + |\mathbf{A}|^2\right)^2\right).$$

since u_1, u_2, u_3, u_4, u_5 are quadratic in $\mathbf{A}$, $\overline{\mathbf{A}}$, and $\tilde{u}_6$ is of order four in $\mathbf{A}$, $\overline{\mathbf{A}}$.

It is not our purpose here to show that (5.5) is integrable and to give a complete description of the set of bounded solutions of (5.5). We shall restrict our analysis to only some special cases which fall in the frame of Il'ichev and Kirchgässner (1997b).

I) $\mathbf{A_1 = B_1 = 0}$. In this case the reduced system in normal form is

(5.6)
$$A_{0,x} = iq_0 A_0 + B_0 + iA_0 P\left(\varepsilon, u_1, u_2, u_5\right) + o\left(|\mathbf{A}|^n\right)$$
$$B_{0,x} = iq_0 B_0 + iB_0 P\left(\varepsilon, u_1, u_2, u_5\right) + A_0 Q\left(\varepsilon, u_1, u_2, u_5\right) + o\left(|\mathbf{A}|^n\right)$$
$$C_x = i\omega_0 C + iCR\left(\varepsilon, u_1, u_2, u_5\right) + o\left(|\mathbf{A}|^n\right)$$

and

$$P\left(\varepsilon, u_1, u_2, u_5\right) = p_1\varepsilon + p_2 u_1 + p_3 u_2 + p_6 u_5 + \mathcal{O}\left(\left(|\varepsilon| + |\mathbf{A}|^2\right)^2\right),$$
$$Q\left(\varepsilon, u_1, u_2, u_5\right) = q_1\varepsilon - q_2 u_1 + q_3 u_2 + q_6 u_5 + \mathcal{O}\left(\left(|\varepsilon| + |\mathbf{A}|^2\right)^2\right),$$
$$R\left(\varepsilon, u_1, u_2, u_5\right) = r_1\varepsilon + r_2 u_1 + r_3 u_2 + r_6 u_5 + \mathcal{O}\left(\left(|\varepsilon| + |\mathbf{A}|^2\right)^2\right).$$

Note that the equations for A_1 and B_1 are satisfied at any order.

The analysis of (5.6) follows Il'ichev and Kirchgässner (1997b). The system (5.6) is integrable and possesses the first integrals

$$K = \frac{i}{2}\left(A_0 \overline{B}_0 - \overline{A}_0 B_0\right), \quad K_1 = C\overline{C}, \quad H = |B_0|^2 - G\left(\varepsilon, |A_0^2|, K, K_1\right),$$

where

$$G = \int_0^{|A_0|^2} Q\left(\varepsilon, u, K, K_1\right) du.$$

Set

$$A_0 = r_0 \exp i\left(q_0 x + \theta_0\right), \quad B_0 = r_1 \exp i\left(q_0 x + \theta_1\right), \quad C = r_2 \exp i\left(\omega_0 x + \theta_2\right).$$

Then (5.6) reads

(5.7)
$$\left(\frac{du_0}{dx}\right)^2 = 4\left\{u_0\left[G\left(\varepsilon, u_0, K, K_1\right) + H\right] - K^2\right\}$$
$$\frac{d\left(\theta_1 - \theta_0\right)}{dx} = -K\left(u_0 u_1\right)^{-1}\left[u_0 Q\left(\varepsilon, u_0, K, K_1\right) + G\left(\varepsilon, u_0, K, K_1\right) + H\right]$$
$$\frac{dr_2}{dx} = 0, \quad \frac{d\theta_2}{dx} = R\left(\varepsilon, u_0, K, K_1\right)$$

where $u_0 = r_0^2$, $u_1 = r_1^2$. The steady solutions of (5.7) are contained in the set determined by the double roots of the polynomial

(5.8)
$$f\left(u_0\right) = u_0\left[G\left(\mu, u_0, K, K_1\right) + H\right] - K^2.$$

Scaling as

$$H = \varepsilon^2 h, \quad u_0 = |\varepsilon| v, \quad K = |\varepsilon|^{3/2} k, \quad K_1 = |\varepsilon| k_1,$$

we obtain

$$f = |\varepsilon|^3 g(v, h, k, k_1) + O(|\varepsilon|^{7/2}),$$

where

$$g(v, h, k, k_1) = vh + (q_1 \operatorname{sign} \varepsilon + q_6 k_1) v^2 - \frac{q_2}{2} v^3 - k^2.$$

The double roots of (5.8) are thus given to the required order by the double roots of the polynomial $g(v, h, k, k_1)$. They lie on curves in the (h, k) plane, given parametrically by

$$k = \pm \sqrt{q_2 v^3 - (q_1 \operatorname{sign} \varepsilon + q_6 k^1) v^2}, \quad h = \frac{3}{2} q_2 v^2 - 2 (q_1 \operatorname{sign} \varepsilon + q_6 k_1) v.$$

Hence the behaviour of the bounded solutions of (5.6) at lowest order in ε are determined by the coefficients q_1, q_2 and q_6, more precisely by the sign of the expression $q_1 \operatorname{sign} \varepsilon + q_6 k_1$. Following the method in Appendix C we obtain

$$q_1 = \frac{1}{4(1 - 3q_0^2)} < 0, \quad q^2 = \frac{1}{96 q_0^2 (3q_0^2 - 1)(5q_0^2 - 1)} > 0$$

$$q_6 = \frac{-2l^2 (2l^2 \Omega (q_0 - \omega_0)^2 + 2l^2 \Omega_1 (\omega_0 + q_0)^2 + \pi^2 (\omega_0^2 + q_0^2))}{(3q_0^2 - 1)(4l^2 \Omega + \pi^2)(4l^2 \Omega_1 + \pi^2)} < 0$$

where

$$\Omega = (\omega_0 + q_0)^2 [(\omega_0 + q_0)^4 - (\omega_0 + q_0)^2 - V_0],$$

and

$$\Omega_1 = (q_0 - \omega_0)^2 [(q_0 - \omega_0)^4 - (q_0 - \omega_0)^2 - V_0].$$

Of course we obtain similar results in the case $A_0 = B_0 = 0$.

II) $A_0 = A_1$, $B_0 = B_1$. This choice is possible since, due to the reflection symmetry S^*, the equations for A_1, B_1 coincide in this case with those for A_0, B_0. From (5.1) and the explicit form of the generalized eigenvectors ϕ_j, ψ_j, we deduce that the solutions found now are even in y. The normal form of the reduced system is as in case I). Moreover, the coefficients q_1 and q_6 are the same, and for q_2 we obtain

(5.9)
$$q_2 = \frac{43 q_0^2 - 8}{96 q_0^2 (3q_0^2 - 1)(5q_0^2 - 1)(13q_0^2 - 2)} > 0.$$

Remark 5.1. – From the formulae for the coefficients q_1, q_2, q_6 in the two particular cases above we can easily deduce the coefficients q_1, q_2, q_4 and q_6 in the complete normal form. These coefficients should determine the behaviour of the bounded solutions in the general case. The coefficients q_1, q_2, q_6 are those from the case I), and for q_4 we find

$$q_4 = q_2 - q_2^{II)} = -\frac{1}{16 q_0^2 (3q_0^2 - 1)(13q_0^2 - 2)}.$$

5.3. Shape of solutions in the case I

We consider here only the case $\operatorname{sign} q_1 + q_6 k_1 > 0$, or $\varepsilon < 0$, $0 \le k_1 < q_1/q_6$, when solitary wave-like structures exist (see Il'ichev and Kirchgässner (1997b) for a complete description of the set of bounded solutions). Then all bounded solutions live in the interior of the curve $g(v, h, k) = 0$, in the (h, k)-plane (see *Fig.* 5 in Iooss and Péroueme, 1993). The point $(0, 0)$ corresponds to a solution homoclinic to a periodic one

$$u_0 = \frac{2\kappa}{q_2} \operatorname{sech}^2 (\sqrt{\kappa} x) + O(|\varepsilon|^{3/2}), \quad \kappa = q_1 \varepsilon + q_6 K_1.$$

$$C = |\varepsilon|^{1/2} k_1 \exp(i\omega_0 x) + O(\varepsilon).$$

or

(5.10) $$A_0 = \pm \sqrt{\frac{2\kappa}{q_2}} \operatorname{sech}(\sqrt{\kappa} x) \exp(i q_0 x) + O(|\varepsilon|^{3/2}).$$

(5.11) $$C = |\varepsilon|^{1/2} k_1 \exp(i\omega_0 x) + O(\varepsilon).$$

Substitution of these expressions in (5.1) yields the formula for the surface deviation η,

(5.12) $$\eta(x, y) = \pm \sqrt{\frac{2\kappa}{q_2}} \operatorname{sech}(\sqrt{\kappa} x) \cos\left(q_0 x + \frac{\pi y}{2l}\right) + 2|\varepsilon|^{1/2} k_1 \cos(\omega_0 x) + O(\varepsilon).$$

These waves exist for ε sufficiently small and a result similar to Theorem 4.2 can be stated.
They are asymmetric waves, asymptotic, as $x \to \infty$, to the periodic wave $2|\varepsilon|^{1/2} k_1 \cos(\omega_0 x) + O(\varepsilon)$. If $k_1 = 0$ they are asymptotic to 0. The shape of the surface deviation η for $k_1 \neq 0$ is showed in Figure 8.

5.4. Shape of solutions in the case II

The solution set of the normal form is in this case similar to that in case I). We find again a one parameter family of solutions homoclinic to a periodic one, given also by (5.10)-(5.11) but with q_2 from (5.9). However, the shape of the surface deviation η is different, we obtain now a symmetric wave (*Fig.* 9). We find

(5.13) $$\eta(x, y) = \pm \sqrt{\frac{2\kappa}{q_2}} \operatorname{sech}(\sqrt{\kappa} x) \cos(q_0 x) \cos\left(\frac{\pi y}{2l}\right) + 2|\varepsilon|^{1/2} k_1 \cos(\omega_0 x) + O(\varepsilon).$$
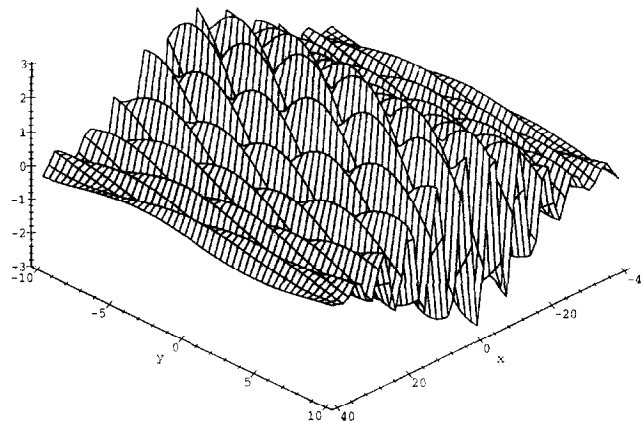
so this solution is even in y.

Fig. 8. – Asymmetric generalized solitary wave of elevation for $q_0 = \sqrt{0.67}$, $k_1 = 0.5$, $\varepsilon = -0.03$.

As in the previous case, these waves are asymptotic, as $x \to \infty$, to a periodic wave.
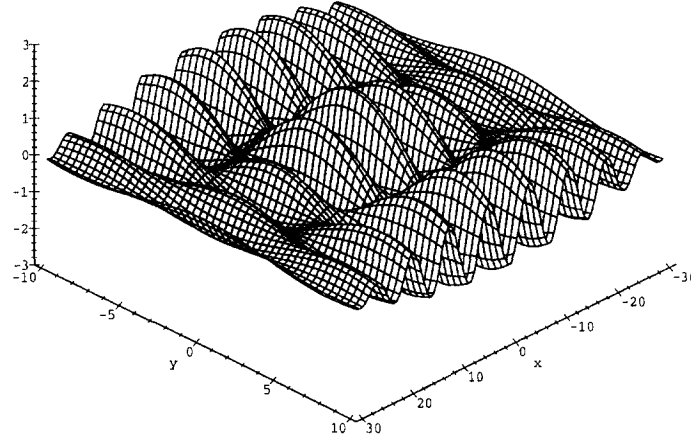

Fig. 9. – Symmetric generalized solitary wave of elevation for $q_0 = \sqrt{0.67}$, $k_1 = 0.5$, $\varepsilon = -0.03$.

6. Further discussion

In this paper we have shown that the model equation (1.1) has non-trivial families of 3D wave solutions which are periodic in the direction transverse to the propagation, and are either decaying, periodic or quasi-periodic in the direction of wave propagation. The existence of steady patterns which are localized in the y-axis direction and periodic in the x-axis direction (direction of wave propagation), which is not considered here, is also of interest. By using again spatial dynamics and center manifold theory one can prove the existence of a family of such solutions (LP family). There exists a wave length close to which the x-periodic waves are modulationally unstable. This wave length corresponds to the wave number where the linear spectrum of the 2D FKdV equation has a minimum for the phase speed. The modulational instability in question implies the formation of a non-steady wave, having the form of an envelope solitary wave in the x-direction and localized in the y-direction from the solution in the LP family. These results are left to a further contribution.

As shown in Section 2 the equation (1.1) is a model equation for several physical phenomena. It is then natural to ask about the existence of the solutions found here in the full problems. The basic domain of these solutions is still a cylindrical one, so the same reduction method seems adapted for this purpose. Moreover, in Section 3 we saw that there is a full correspondence between the critical spectra in the model equation and in the Euler system for water waves. So, both systems are expected to have the same asymptotic properties near equilibrium. However, the main difficulty in treating the full water wave problem consists in justifying the reduction. We are still not able to find a correct dynamical formulation for (2.1) which allows us to apply the reduction procedure by Mielke (1988a).

APPENDIX A: Resolvent estimates

Proof of Lemma 4.1. – Choose $\hat{q} > q_0$, so that for any $q > \hat{q}$, iq belongs to the resolvent set of $\mathcal{A}$, i.e. the linear operator $(\mathcal{A} - iq)^{-1}$ exists.

Assume $\mathbf{f} \in W$, so $\mathbf{f} = (0,0,0,0,0,f)^t$, and $\eta = (\mathcal{A} - iq)^{-1}\mathbf{f}$, $\eta = (\eta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5)^t$. We show

(A.1)
$$\|\eta\|_{X_l} \leq \frac{C}{|q|}\|\mathbf{f}\|_{X_l},$$

for any $q > \hat{q}$.

Denote by $\|\cdot\|_s$ the usual Sobolev norm in $H^s(-l, l)$.

The vectors η and $\mathbf{f}$ satisfy the system

(A.2)
$$\eta_1 - iq\eta = 0$$
$$\eta_2 - iq\eta_1 = 0$$
$$\eta_3 - iq\eta_2 = 0$$
$$\eta_4 - iq\eta_3 = 0$$
$$\eta_5 - iq\eta_4 = 0$$
$$-\eta_{yy} + V_0\eta_2 - \eta_4 - iq\eta_5 = f.$$

Recall that $\mathcal{X}_l = H_0^{5/3} \times H_0^{4/3} \times H_0^1 \times H_0^{2/3} \times H^{1/3} \times H^0$, so we have to estimate $\eta \in H^{5/3}$, $\eta_1 \in H^{4/3}$, $\eta_2 \in H^1$, $\eta_3 \in H^{2/3}$, $\eta_4 \in H^{1/3}$, $\eta_5 \in H^0$, when $f \in H^0$.

The system above yields

$$-\eta_{yy} - V_0 q^2 \eta - q^4 \eta + q^6 \eta = f,$$

so

$$\|\eta\|_0 \leq \frac{C}{|q|^6}\|f\|_0, \quad \|\eta\|_2 \leq C\|f\|_0.$$

By using interpolation inequalities for Sobolev spaces we deduce

$$\|\eta\|_{j/3} \leq \frac{C}{|q|^{6-j}}, \quad j = 0, \ldots 6.$$

Then, these inequalities together with the system (A.2) yield (A.1). □

APPENDIX B: Normal form analysis

In this Appendix we compute the normal form of a nonlinear system of ordinary differential equations

(B.1)
$$\frac{d}{dx}\mathbf{A} = \mathcal{L}\mathbf{A} + \mathcal{F}(\mathbf{A}),$$

with $\mathbf{A} = (A_0, B_0, A_1, B_1, C, \overline{A}_0, \overline{B}_0, \overline{A}_1, \overline{B}_1, \overline{C})$, and linear part

$$\mathcal{L} = \begin{pmatrix} J & 0 & 0 & 0 & 0 & 0 \\ 0 & J & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{J} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{J} & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\omega_0 \end{pmatrix}, \quad J = \begin{pmatrix} iq_0 & 1 \\ 0 & iq_0 \end{pmatrix}.$$

We assume that the system is invariant under the action of the symmetries τ_a^*, S^* and anticommutes with the reversibility operator R^* in Section 5.2.

From the general theory of normal forms (see e.g. Elphick *et al.*, 1987, for a characterization at any order) we know that there exists a change of variables which is close to identity, and transforms the system (B.1) into

$$(B.2) \qquad \frac{d}{dx}\,\mathbf{A} = \mathcal{L}\mathbf{A} + \mathcal{P}\,(\mathbf{A}) + o\,(\|\mathbf{A}\|^n),$$

where $\mathcal{P}$ is a polynomial of degree $\leq n$, with $\mathcal{P}\,(0) = 0$, and $D\mathcal{P}\,(0) = 0$. Moreover, $\mathcal{P}$ satisfies the equality

$$(B.3) \qquad D\mathcal{P}\,(\mathbf{A})\,\mathcal{L}^*\mathbf{A} = \mathcal{L}^*\mathcal{P}(\mathbf{A}),$$

for any $\mathbf{A}$.

Let $\mathcal{P} = (P_0, Q_0, P_1, Q_1, R, \overline{P}_0, \overline{Q}_0, \overline{P}_1, \overline{Q}_1, \overline{R})^t$, and define the differential operator

$$\mathcal{D}^* = -iq_0 A_0\,\frac{\partial}{\partial A_0} + (A_0 - iq_0 B_0)\,\frac{\partial}{\partial B_0} - iq_0 A_1\,\frac{\partial}{\partial A_A} + (A_1 - iq_0 B_1)\,\frac{\partial}{\partial B_1} - i\omega_0\,\frac{\partial}{\partial C} + c.c.$$

Then (B.3) is equivalent with

$$\mathcal{D}^*\mathcal{P} = \mathcal{L}^*\mathcal{P},$$

or explicitely

$$(B.4) \qquad \mathcal{D}^* P_0 = -iq_0 P_0, \quad \mathcal{D}^* Q_0 = -iq_0 Q_0 + P_0,$$

$$(B.5) \qquad \mathcal{D}^* P_1 = -iq_0 P_1, \quad \mathcal{D}^* Q_1 = -iq_0 Q_1 + P_1, \quad \mathcal{D}^* R = -i\omega_0 R.$$

In order to determined $\mathcal{P}$ we need nine independent first integrals of $\mathcal{D}^* = 0$. We find the following first integrals:

$$u_1 = A_0 \overline{A}_0, \quad u_2 = \frac{i}{2}\,(A_0 \overline{B}_0 - \overline{A}_0 B_0), \quad u_3 = A_1 \overline{A}_1, \quad u_4 = \frac{i}{2}\,(A_1 \overline{B}_1 - \overline{A}_1 B_1),$$

$$u_5 = C\overline{C}, \quad u_6 = A_0 \overline{B}_1 - \overline{A}_1 B_0, \quad u_7 = A_0 \overline{A}_1 \quad (\text{or } \tilde{u}_7 = \overline{A}_0 A_1).$$

$$u_8 = \frac{B_0}{A_0} + \frac{1}{iq_0}\,\ln A_0, \quad u_9 = \frac{B_0}{A_0} + \frac{1}{i\omega_0}\,\ln C.$$

LEMMA 6.1. – *Assume P is a polynomial of the components of $\mathbf{A}$ and $\mathcal{D}^* P = 0$. If P is invariant under the action of τ_a^* then $P = Q\,(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6)$, where Q is a polynomial and $\tilde{u}_6 = iu_6 \tilde{u}_7/2$.*

Proof. – From the general theory of differential operators follows that $P = Q\,(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9)$. We have to show that Q is a polynomial and it is independent of u_8, u_9.

Since Q is invariant under τ_a^*, for any $a \in \mathbb{R}$, the expression

$$Q\left(u_1, u_2, u_3, u_4, u_5, e^{2i\phi} u_6, e^{2i\phi} u_7, \frac{\phi}{q_0} + u_8, u_9\right)$$

is independent of $\phi = \pi a/2l$. By differentiating it with respect to ϕ we find that

$$(B.6) \qquad 2ie^{2i\phi}\left(u_6\,\frac{\partial Q}{\partial u_6} + u_7\,\frac{\partial Q}{\partial u_7}\right) + \frac{1}{q_0}\,\frac{\partial Q}{\partial u_8} = 0,$$

holds for any ϕ. In particular,

$$\frac{\partial Q}{\partial u_8} = 0.$$

so Q does not depend on u_8.

In order to show that Q is a polynomial in its arguments we look at the relation between the partial derivatives of P and Q. We find

(B.7)
$$\frac{\partial P}{\partial \overline{A}_0} = A_0 \frac{\partial Q}{\partial u_1} - \frac{i}{2} B_0 \frac{\partial Q}{\partial u_2}$$

(B.8)
$$\frac{\partial P}{\partial B_0} = -\frac{i}{2} \overline{A}_0 \frac{\partial Q}{\partial u_2} - \overline{A}_1 \frac{\partial Q}{\partial u_6} + \frac{1}{A_0} \frac{\partial Q}{\partial u_9}$$

(B.9)
$$\frac{\partial P}{\partial \overline{B}_0} = \frac{i}{2} A_0 \frac{\partial Q}{\partial u_2}$$

(B.10)
$$\frac{\partial P}{\partial A_1} = \overline{A}_1 \frac{\partial Q}{\partial u_3} + \frac{i}{2} \overline{B}_1 \frac{\partial Q}{\partial u_4}$$

(B.11)
$$\frac{\partial P}{\partial \overline{A}_1} = A_1 \frac{\partial Q}{\partial u_3} - \frac{i}{2} B_1 \frac{\partial Q}{\partial u_4} - B_0 \frac{\partial Q}{\partial u_6} + A_0 \frac{\partial Q}{\partial u_7}$$

(B.12)
$$\frac{\partial P}{\partial B_1} = -\frac{i}{2} \overline{A}_1 \frac{\partial Q}{\partial u_4}$$

(B.13)
$$\frac{\partial P}{\partial \overline{B}_1} = \frac{i}{2} A_1 \frac{\partial Q}{\partial u_4} + A_0 \frac{\partial Q}{\partial u_6}$$

(B.14)
$$\frac{\partial P}{\partial C} = \overline{C} \frac{\partial Q}{\partial u_5} + \frac{1}{i\omega_0 C} \frac{\partial Q}{\partial u_9}$$

(B.15)
$$\frac{\partial P}{\partial \overline{C}} = C \frac{\partial Q}{\partial u_5}.$$

From (B.9), (B.12) and (B.15) follows first that Q is a polynomial in u_2, u_4 and u_5. Then from (B.7), (B.10) and (B.13) we obtain that Q is a polynomial in u_1, u_3 and u_6. Finally, from (B.8) it follows that Q is a polynomial in u_9.

Since Q is polynomial in u_9, by comparing the asymptotic behaviour of P and Q as $C \to \infty$ we deduce that in fact they should be independent of u_9. Hence $P = Q(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$. Now, since Q is a polynomial in the first six arguments and u_j is a polynomial of the components of $\mathbf{A}$, for $j = \overline{1, 7}$, it follows that Q is a rational function in u_7. Hence

(B.16)
$$Q(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = \sum q_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6}(u_7) u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} u_4^{\alpha_4} u_5^{\alpha_5} u_6^{\alpha_6},$$

and $q_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}(u_7)$ is a rational function of u_7. By looking at the definition of u_j we see that if $\alpha_1\alpha_3 = 0$ the coefficient $q_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}(u_7)$ has to be a polynomial in u_7. If $\alpha_1\alpha_3 \neq 0$, then it might be of the form

$$q_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}(u_7) = \frac{p_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}}{u_7^{\alpha_7}}.$$

where $\alpha_7 \leq \min\{\alpha_1, \alpha_3\}$. Substitution of this relation into (B.16) together with (B.6) imply $\alpha_6 = \alpha_7 > 0$. Then

$$q_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}(u_7)\, u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} u_4^{\alpha_4} u_5^{\alpha_5} u_6^{\alpha_6} = p_{\alpha_1'\alpha_2\alpha_3'\alpha_4\alpha_5\alpha_7}\, u_1^{\alpha_1'} u_2^{\alpha_2} u_3^{\alpha_3'} u_4^{\alpha_4} u_5^{\alpha_5} u_6^{\alpha_7} \tilde{u}_7^{\alpha_7}.$$

since $\tilde{u}_7 u_7 = u_1 u_3$, where $\alpha_1' = \alpha_1 - \alpha_7 > 0$, $\alpha_3' = \alpha_3 - \alpha_7 > 0$. $\square$

We start now to calculate the components of the polynomial $\mathcal{P}$ in the normal form of (B.1).

From (B.4) and the definition of $\mathcal{D}^*$ we obtain $\mathcal{D}^* = (\overline{A}_0 P_0) = 0$. Moreover, $\overline{A}_0 P_0$ is invariant under the action of τ_a^* so by the lemma above

$$\overline{A}_0 P_0 = P_0^*(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6),$$

with P_0^* polynomial in its arguments. Since $P_0^*/\overline{A}_0$ is also a polynomial we deduce

$$P_0 = A_0 \breve{P}_0(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6) + A_1 u_6 \hat{P}_0(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6).$$

A direct calculation shows that a particular solution of $\mathcal{D}^* Q_0 = -iq_0 Q_0 + P_0$ is

$$Q_0^* = B_0 \breve{P}_0 + \frac{A_1 B_0}{A_0} u_6 \hat{P}_0.$$

Since Q_0^* is also a polynomial in the components of $\mathbf{A}$ from the equality above we deduce that the last term in the right hand side is a polynomial, so

$$\frac{A_1 B_0}{A_0} u_6 \hat{P}_0 = A_1 B_0 \overline{A}_0 u_6 \hat{P}_0^*(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6).$$

Then

$$P_0 = A_0 \breve{P}_0 + A_1 u_6 \hat{P}_0 = A_0 \breve{P}_0 + A_1 A_0 \overline{A}_0 u_6 \hat{P}_0^* = A_0 \tilde{P}_0(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6).$$

A particular solution of $\mathcal{D}^* Q_0 = -iq_0 Q_0 + P_0$ is now $B_0 \tilde{P}_0$, so

$$Q_0 = B_0 \tilde{P}_0(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6) + A_0 \tilde{Q}_0(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6).$$

The next two components P_1 and Q_1 are obtained from the invariance of the normal form under the action of S^*. We deduce

$$P_1 = A_1 \tilde{P}_0(u_3, u_4, u_1, u_2, u_5, \overline{\tilde{u}_6}),$$

and

$$Q_1 = B_1 \tilde{P}_0(u_3, u_4, u_1, u_2, u_5, \overline{\tilde{u}_6}) + A_1 \tilde{Q}_0(u_3, u_4, u_1, u_2, u_5, \overline{\tilde{u}_6}).$$

Finally, $\mathcal{D}^*(\overline{C}R) = 0$ and $\overline{C}R$ is invariant under the action of τ_a^*, so $\overline{C}R = R^*(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6)$, with R^* polynomial in its arguments. Since $R^*/\overline{C}$ is a polynomial in the components of $\mathbf{A}$ we deduce

$$R = \frac{1}{\overline{C}} R^*(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6) = C\tilde{R}(u_1, u_2, u_3, u_4, u_5, \tilde{u}_6),$$

and $\tilde{R}$ is a polynomial.

One can now use the reversibility and deduce that P, R have purely imaginary coefficients and Q has real coefficients, except for the coefficients of the terms containing $\tilde{u}_0$.

APPENDIX C: Coefficients of the normal form

I) $\mathbf{A_1} = \mathbf{B_1} = 0$. In this case any small bounded solution of (4.1) is of the form

(C.1) $\mathbf{w} = A_0\phi_0 + B_0\psi_0 + Cf_0 + \overline{A}_0\overline{\phi}_0 + \overline{B}_0\overline{\psi}_0 + \overline{C}\overline{f}_0 + \Phi\left(\varepsilon, A_0, B_0, C, \overline{A}_0, \overline{B}_0, \overline{C}\right).$

where Φ contains terms which are at least quadratic in its arguments. For the calculation of the coefficients q_1, q_2 and q_6 we follow the method by Dias and Iooss (1993).

As in the case of the 1:1 resonance (Section 4.2) the coefficient q_1 is related to the eigenvalues σ of the linearization of (5.6) at 0. We find

$$\sigma = iq_0 \pm \sqrt{q_1\varepsilon} + ip_1\varepsilon + O\left(|\varepsilon|^{3/2}\right).$$

In the next calculations we can set $\varepsilon = 0$ since this has no influence on the final result. We write the equation (4.1) (with $\varepsilon = 0$) as

(C.2) $\mathbf{w}_{,r} = \mathcal{A}\mathbf{w} + N_2\left(\mathbf{w}, \mathbf{w}\right) + N_3\left(\mathbf{w}, \mathbf{w}, \mathbf{w}\right),$

where N_2 denotes the quadratic terms of the nonlinear operator $F\left(0, \cdot\right)$ and N_3 the cubic terms. It is easy to see that $N_3 = 0$.

Take

$$\Phi\left(0, A_0, B_0, C, \overline{A}_0, \overline{B}_0, \overline{C}\right) = \left(\Phi_{200000}A_0^2 + c.c.\right) + \Phi_{110000}A_0\overline{A}_0 + \left(\Phi_{000020}C^2 + c.c.\right)$$
$$+ \Phi_{000011}C\overline{C} + \left(\Phi_{100010}A_0C + c.c.\right) + \left(\Phi_{100001}A_0\overline{C} + c.c.\right)$$
$$+ \left(\Phi_{210000}A_0|A_0|^2 + c.c.\right) + \left(\Phi_{1000011}A_0|C|^2 + c.c.\right) + \ldots.$$

Differentiation of (C.1) with respect to x and the formula above yield

(C.3) $\mathbf{w}_{,r} = \left(A_{0,r}\phi_0 + B_{0,r}\psi_0 + C_r f_0 + c.c.\right) + \left(2A_0A_{0,r}\Phi_{200000} + c.c.\right)$
$+ \left(\overline{A}_{0,r}A_0 + \overline{A}_0A_{0,r}\right)\Phi_{110000} + \left(2CC_r\Phi_{000020} + c.c.\right) + \left(\overline{C}_rC + \overline{C}C_r\right)\Phi_{000011}$
$+ \left(\left(A_{0,r}C + A_0C_r\right)\Phi_{100010} + c.c.\right) + \left(\left(A_{0,r}\overline{C} + A_0\overline{C}_r\right)\Phi_{100001} + c.c.\right)$
$+ \left(\left(2A_{0,r}|A_0|^2 + A_0^2\overline{A}_{0,r}\right)\Phi_{210000} + c.c.\right)$
$+ \left(\left(A_{0,r}|C|^2 + A_0C_r\overline{C} + A_0C\overline{C}_r\right)\Phi_{100011} + c.c.\right) + \ldots.$

By equating the powers of A_0, B_0, C in the right hand sides of (C.2) and (C.3) we obtain:

$$A_0 : \mathcal{A}\phi_0 = iq_0\phi_0,$$

$$B_0 : \mathcal{A}\psi_0 = i\omega\psi_0 + \phi_0,$$

$$C : \mathcal{A}f_0 = i\omega_0 f_0,$$

$$A_0^2 : \mathcal{A}\Phi_{200000} + N_2\,(\phi_0,\phi_0) = 2iq_0\Phi_{200000},$$

$$A_0\overline{A}_0 : \mathcal{A}\Phi_{110000} + 2N_2\,(\phi_0,\overline{\phi}_0) = 0,$$

$$C^2 : \mathcal{A}\Phi_{000020} + N_2\,(f_0,f_0) = 2i\omega_0\Phi_{000020},$$

$$C\overline{C} : \mathcal{A}\Phi_{000011} + 2N_2\,(f_0,\overline{f}_0) = 0,$$

$$A_0C : \mathcal{A}\Phi_{100010} + 2N_2\,(\phi_0,f_0) = i\,(q_0 + \omega_0)\,\Phi_{100010},$$

$$A_0\overline{C} : \mathcal{A}\Phi_{100001} + 2N_2\,(\phi_0,\overline{f}_0) = i\,(q_0 - \omega_0)\,\Phi_{100001},$$

$$A_0^2\overline{A}_0 : \mathcal{A}\Phi_{210000} + 2N_2\,(\phi_0,\Phi_{110000}) + 2N_2\,(\overline{\phi}_0,\Phi_{200000}) = iq_0\Phi_{210000} + ip_2\phi_0 - q_2\psi_0,$$

$$A_0|C|^2 : \mathcal{A}\Phi_{100011} + 2N_2\,(f_0,\Phi_{100001}) + 2N_2\,(\overline{f}_0,\Phi_{100010}) + 2N_2\,(\phi_0,\Phi_{000011})$$

$$= iq_0\Phi_{100011} + ip_6\phi_0 + q_6\psi_0,$$

Take the scalar product of the right hand sides in the last two equalities with the adjoint eigenvector ψ_0^* ($\mathcal{A}^*\psi_0^* = -iq_0\psi_0^*, \langle\psi_0,\psi_0^*\rangle = 1$) and find

(C.4) $$q_2 = -\langle 2N_2(\phi_0,\Phi_{110000}) + 2N_2(\overline{\phi}_0,\Phi_{200000}),\psi_0^*\rangle,$$

(C.5) $$q_6 = \langle 2N_2(f_0,\Phi_{100001}) + 2N_2(\overline{f}_0,\Phi_{100010}) + 2N_2(\phi_0,\Phi_{000011}),\psi_0^*\rangle.$$

II) $\mathbf{A_0 = A_1}$, $\mathbf{B_0 = B_1}$. In this case the calculations are similar. The only difference is that now

$$\mathbf{w} = A_0\,(\phi_0 + \phi_1) + B_0\,(\psi_0 + \psi_1) + Cf_0 + \overline{A}_0\,(\overline{\phi}_0 + \overline{\phi}_1) + \overline{B}_0\,(\overline{\psi}_0 + \overline{\psi}_1) + \overline{C}\overline{f}_0 + \Phi\,(\varepsilon, A_0, B_0, C, \overline{A}_0, \overline{B}_0, \overline{C}).$$

So in the calculations above we have to replace ϕ_0 by $\phi_0 + \phi_1$, and ψ_0 by $\psi_0 + \psi_1$. Then the same formula holds for q_1, and

(C.6) $$q_2 = -\langle 2N_2(\phi_0 + \phi_1,\Phi_{110000}) + 2N_2(\overline{\phi}_0 + \overline{\phi}_1,\Phi_{200000}),\psi_0^* + \psi_1^*\rangle,$$

(C.7) $$q_6 = \langle 2N_2(f_0,\Phi_{100001}) + 2N_2(\overline{f}_0,\Phi_{100010}) + 2N_2(\phi_0 + \phi_1,\Phi_{000011}),\psi_0^* + \psi_1^*\rangle.$$

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